"HODGE THEORY": PROGRAM

INTRODUCTION

A sample of results related to Hodge theory: topological restrictions on algebraic varieties, hyperbolicity results, results on algebraic cycles. The motivating example: periods of abelian differentials on a compact Riemann surface.

THE CLASSICAL THEORY OF KÄHLER VARIETIES

Hodge Theory for Riemannian manifolds: the statements, the operators H and G. Kähler manifolds: equivalent definitions, general properties. Hodge theory for Kähler manifolds: the (p,q)-decomposition, the three operators L, H, Λ. The Kähler identities. The "Hard Lefschetz theorem", the Hodge-Riemann bilinear relations. Cohomology class of a complex subvariety. A specific feature of complex geometry: positivity.

SPECIAL FEATURES OF THE PROJECTIVE CASE

Integral Kähler classes, Kodaira imbedding theorem. A glimpse of vanishing theorems via Hodge theory. The Lefschetz hyperplane section theorem. A proof of the Hard Lefschetz theorem using the Hodge-Riemann bilinear relations and Lefschetz pencils. The algebraic de Rham theorem. Short discussion of absolute Hodge theory.

MIXED HODGE THEORY

The yoga of weights: some examples: Serre's remark ("Analogues Kähleriens de certains conjectures de Weil"). A guiding example: The Hodge structure of a nodal curve. Definition of Mixed Hodge structure. Strictness of maps between MHS's. The category of MHS is abelian.
The two main examples:

- Mixed Hodge structure on the cohomology of a quasi-projective nonsingular variety.
- Mixed Hodge structure on the cohomology of a normal crossing variety.

Sketch of the construction of a MHS on the cohomology of a quasi-projective variety in the general case. Restrictions on weights.

Main consequences of Mixed Hodge theory: the weight trick, the additivity of the weight polynomial. The global invariant cycle theorem, the semisimplicity of monodromy representations. A third proof of Hard Lefschetz using semisimplicity of monodromy and Lefschetz pencils.

VARIATIONS OF HODGE STRUCTURES

Motivation: the Siegel upper half-space and the period map for curves. Polarizations. Polarized Weight one PHS's and Abelian varieties (Riemann's characterization of Abelian varieties).
The geometry of period domains. Variations of pure Hodge structures. Some properties of the period map: Griffiths' transversality, curvature estimates. Schmid's paper: The nilpotent orbit theorem. Quasi-unipotency of local monodromy. The limit Mixed Hodge structure associated to a degenerating family.

TOWARDS NON-ABELIAN HODGE THEORY (if time permits)

Classical Hodge theory and GL(1) representations of the fundamental group. The Betti, de Rham, Dolbeault model in the abelian case. Some differential geometry of complex vector bundles. The theorem of Narasimhan and Seshadri on unitary representations of the fundamental group af a Riemann surface.
Non unitary representations and Higgs fields. Hitchin's work on non-abelian Hodge theory for Riemann surfaces.

REFERENCES

E. Cattani et al. (editors) "Hodge theory," Mathematical Notes 49, Princeton University Press 2014, Chapters 1,4,7.

P. Deligne "Travaux de Griffiths" Séminaire Bourbaki 376.

P. Deligne, "Théorie de Hodge, II," Publ. Math. IHES 40 (1971), 5-57.

P. Deligne, "Théorie de Hodge, III," Publ. Math. IHES 44 (1974), 5-78.

P. Griffiths (editor) "Topics in transcendental Algebraic geometry", Chapters I,II,IV,VI.

Annals of Mathematics studies 106, Princeton University Press 1984.

J. Le Potier "Fibrés de Higgs et systèmes locaux" Séminaire Bourbaki 737

C. Peters J. Steenbrink "Mixed Hodge structures," Ergebn. der Math. 52, Springer

W. Schmid "Variation of Hodge structure: the singularities of the period mapping", Invent. Math. 22, p.211-319 (1973)

C.Voisin, "Hodge theory and complex algebraic geometry, I, II, Cambridge St. in Adv. Math. 76, 77. Cambridge University Press 2003.