Lectures on analytic linear systems

The course is aimed at exposing the modern theory of linear ordinary differential equations in the complex domain. Although the subject is utterly classical and some of the main results were achieved in mid-19th century, the traditional approach in the classical sources (Ince, Hille e.a.) is based on heavy computations. The geometric style of exposition allows to reduce the computations to a bare minimum while stressing the geometric nature of the discussed phenomena.

On the other hand, some parts of the course will be devoted to relatively recent achievements. E.g., only several years ago A. Bolibruch discovered a very subtle obstruction to solvability of the Riemann–Hilbert problem. I plan to describe the obstruction in terms of meromorphic connexions on the holomorphic bundles over the Riemann sphere.

The preliminary program is as follows.

General facts about linear systems

— Linear differential equations: Pfaffian, ordinary, matrix
— Fundamental solutions
— Monodromy and holonomy
— Gauge transform and gauge equivalence
— Systems with isolated singularities

Local theory of regular singular points and applications

— Regular singularities
— Fuchsian singularities
— Formal classification of Fuchsian singularities
— Holomorphic classification of Fuchsian singularities
— Integrability of the normal form
— Further simplification of the normal form for Fuchsian systems
— Nonlocal theory of linear systems on $\mathbb{P}$: the Riemann-Fuchs theorem and the Riemann–Hilbert problem
— Monopoles

Global theory of linear systems: holomorphic vector bundles and meromorphic connexions

— Holomorphic vector bundles
— Cocycles

— Operations on bundles
— Classification of line bundles over the Riemann sphere
— Sections of holomorphic vector bundles
— Degree of a holomorphic bundle
— Holomorphic and meromorphic connexions
— Connexions vs. linear systems
— Connexions on line bundles. Trace of a meromorphic connexion
— Classification of holomorphic vector bundles over $\mathbb{P}$

Riemann–Hilbert problem

— Riemann–Hilbert problem on abstract bundles
— Connexions on the trivial bundle
— Invariant subbundles and irreducibility
— Bolibruch–Kostov theorem
— Bolibruch counterexample

Linear nth order differential equations

— High order differential operators: algebraic theory
— Linear ordinary differential equations: the naïve approach
— Factorization of differential operators
— Fuchsian singularities of nth order equation
— Jet bundles and invariant constructions
— Riemann–Hilbert problem for higher order equations

Irregular singularities and the Stokes phenomenon

— One-dimensional irregular singular points
— Birkhoff standard form
— Resonances and formal diagonalization
— Formal simplification in the resonant case
— Shearing transformations and ramified formal normal form
— Holomorphic sectorial normalization
— Sectorial automorphisms and Stokes matrices
— Stokes phenomenon. Holomorphic classification of irregular singularities
— Realization theorem