The ghosts of the Ecole Normale

Life, death and legacy of René Gateaux[1]

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Abstract

The present paper deals with the life and some aspects of the scientific contributions of the mathematician René Gateaux, killed during World War 1 at the age of 25. Though he died very young, he left interesting results in functional analysis. In particular, he was among the first to try to construct an integral over an infinite dimensional space. His ideas were extensively developed later by Paul Lévy. Among other things, Lévy interpreted Gateaux's integral in a probabilistic framework that later contributed to the construction of the Wiener measure. This article tries to explain this singular personal and professional destiny in pre- and postwar France.

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Introduction

In his 1923 seminal paper on Brownian motion, Norbert Wiener mentioned[3] that *integration in infinitely many dimensions [was] a relatively little-studied problem* and that *all that has been done on it [was] due to Gateaux, Lévy, Daniell* and himself. Following Wiener, the most complete investigations had been those *begun by Gateaux and carried out by Lévy*.

It was in 1922 that Lévy's book *Leçons d'Analyse Fonctionnelle* ([Lévy, 1922]) was published after his lectures given at the Collège de France in the aftermath of the Great War. Lévy's book, and more specifically Lévy himself, made a profound impression on Wiener [Wiener, 1923] (p.132). The American mathematician emphasized how Lévy explained personally to him how his own method of integration in infinitely many dimensions, which extended results Lévy found in Gateaux's works, was the convenient tool he needed for his construction of Brownian motion measure.

I shall comment later on the path linking Gateaux's works to Lévy's fundamental studies, but let me begin by discussing the circumstances which constituted the initial motivation behind the current paper. Gateaux was killed at the very beginning of the Great War in October of 1914. He died at the age of 25, before having obtained any academic position, even before having completed a doctorate. His publications formed a rather thin set of a few notes presented to the Academy of

[1] In all the literature, there is a significant uncertainty regarding whether the name bears a circumflex accent or not (due to the confusion with the word *gâteau* -cake in French). In the present paper, I shall adopt the mathematician's own use of *NOT* writing the name with an accent (this is to conform with his birth certificate).

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[3] [Wiener, 1923], p.132.

Sciences of Paris and to the Accademia dei Lincei of Rome. None of them dealt with infinite-dimensional integration. Neverthess, Gateaux's name is still known today, and even to (some) undergraduate students through a basic notion of calculus known as the *Gateaux differentiability*[4]. The notion, weaker than the (now) classical Fréchet differentiability, was mentioned in Gateaux's note [Gateaux, 1914a] (p.311), under the name *variation première* of a functional, though it was probably already considered by him in 1913 as the name appears in [Gateaux, 1913a] (p.326) but without any definition. Regardless, this notion was in fact only a technicality introduced by Gateaux among the general properties that a functional can have. Lévy was probably the first to name it after Gateaux[5].

So, I wanted to understand how it can be explained that a basic notion of calculus had been given the name of an unknown mathematician, who died so young, before having obtained any academic position and even before having defended a thesis. Such a paradox deserved to be unravelled. It is this apparent contradiction that I want to address in this paper by presenting René Gateaux's life and death, some of his mathematical research and the path explaining why we still remember him though so many of his fellows killed during the war became only a *golden word on our public squares* following Aragon's beautiful expression [6].

Let me immediately reveal the key to our explanation. Beyond his tragic fate, Gateaux had two strokes of good fortune. The first one was related to the main mathematical theme he was interested in, *Functional Analysis* (Analyse Fonctionnelle) in the spirit of Volterra in Rome and Hadamard in Paris, often also called by them *functional calculus* (calcul fonctionnel)[7]. At the beginning of the 20th Century, this subject was still little studied. In the years following World War 1, it received unexpected developments, in particular in the unpredictable direction of probability theory. Gateaux was therefore posthumously in contact with a powerful stream leading to the emergence of some central aspects of modern probability, such as Brownian motion as we have seen in Wiener's own words. It is very fortunate for the historian that important archival documents about Gateaux's beginnings in mathematics are still available. Gateaux had in particular been in correspondence with Volterra before, during, and (for some weeks) after a sojourn in Rome with the Italian mathematician. His letters still exist today at the *Accademia dei Lincei* and provide precious insight into Gateaux's first steps. Letters exchanged between Borel and Volterra about the young man's projects and progress are also available. One such document is a letter from Gateaux to Volterra dated from 25 August 1914 and written on the battlefield. Moreover, some other material is accessible such as the military dossier, some of Gateaux's own drafts of reports about his work, and some scattered letters from or about him by other people. This allows us to attempt to reconstruct the life of the young mathematician during his last seven or eight years.

[4]Let me recall that Gateaux differentiability of a function f defined on $I\!R^n$ is the directional differentiability : f is said Gateaux differentiable at $x \in I\!R^n$ if for any vector u given in $I\!R^n$, the function $\lambda \mapsto f(x + \lambda u)$ has a derivative at 0.

[5]In [Lévy, 1922], p.51, under the name *différentielle au sens de Gateaux*. Sanger ([Sanger, 1933]) compared the various definitions formulated for the differential of a functional in his survey about Volterra's functions of lines (See in particular Chapter II on pp.240-253. Gateaux's definition is considered on pp.250-251.).

[6]Déjà la pierre pense où votre nom s'inscrit/ Déjà vous n'êtes plus qu'un mot d'or sur nos places/ Déjà le souvenir de vos amours s'efface/Déjà vous n'êtes plus que pour avoir péri ([Aragon, 1956])

[7]In the sequel, I shall use the expression *functional analysis* only in reference to the theories initiated by Volterra, though it today has a slightly different meaning.

But it is mainly due to the second stroke of fortune that some memory of Gateaux (or, at least, of his name) has been preserved. Before he went to the war, Gateaux had left his papers in his mother's house. Among them were several half-completed manuscripts which were intended to become chapters of his thesis. After the death of her son, his mother sent the papers to the Ecole Normale. Hadamard collected them and in 1919, passed them to Paul Lévy in order to prepare an edition in Gateaux's honor. Studying Gateaux's papers came at a crucial moment in Lévy's career. Not only did they inspire Levy's book [Lévy, 1922] but they were a major source for his later achievements in probability theory.

The aim of the present paper is twofold : one aspect is to present an account of Gateaux's life by using valuable new archival material discovered in several places, the other is to give some hints of how his works were completed and - considerably - extended by Lévy. In that respect, it is clear that the mathematical ideas of Gateaux were developed in a direction he could not have expected ; probability, for instance, was absolutely not in his mind. The appearance of the mathematics of randomness in this inheritance is undoubtedly entirely due to Lévy's powerful imagination. It is therefore well beyond the scope of this article, centered on Gateaux, to present a detailed study of Lévy or Wiener's studies on Brownian motion. The interested reader may refer to several historical expositions such as [Kahane, 1998] or [Barbut, 2004] (pp.54-60). An account from direct participants in this story can be found in Lévy's autobiography [Lévy, 1970] (p.96 and seq.) or Itô's comments on Wiener's papers ([Wiener, 1976], pp.513-519).

Looking backward, Gateaux's role must not be overestimated in the history of mathematics. Contrary to some other examples of mathematicians who died young, such as Abel to cite a famous example, Gateaux had not made decisive progress in any important direction. So maybe some words are necessary to explain what a biographical approach of someone as Gateaux can teach us. The main point here is related to the Great War, and the effect it produced on French mathematicians.

In her memoirs [Marbo, 1967], written at the end of the 1960's, the novelist Camille Marbo[8], Emile Borel's widow, mentioned that after the end of World War 1, her husband declared that he could not bear any more the atmosphere of the Ecole Normale in mourning, and decided to resign from his position of Deputy Director. In 1910 Borel had succeeded the position to Jules Tannery, during a time of extraordinary success for Analysis in France with outstanding mathematicians as Henri Poincaré, Emile Picard, Jacques Hadamard, Henri Lebesgue, and naturally Borel himself.

A superficial, though impressive picture of the effect of WW1 on the French mathematical community is read through the personal life of the aforementioned mathematicians - with the obvious exception of Poincaré who had died in 1912 . Picard lost one son in 1915, Hadamard two sons in 1916 (one in May, one in July) and Borel his adopted son in 1915. The figures concerning casualties among the students of the Ecole Normale, and especially among those who had just finished their three year studies at the *rue d'Ulm*, are terrible[9]. Out of about 280 pupils who entered the Ecole Normale in the years 1911 to 1914, 241 were sent to the front directly from the school and 101 died during the war. If the President of the Republic Raymond Poincaré could declare that *the Ecole of 1914 has avenged the Ecole of 1870*[10], the price to pay had been so enormous that it was difficult to understand how French science could survive such a haemorrhage. Most of the vanished were brilliant young men, expected successors of the brightest scholars from the

[8]Marbo is Marguerite Appel's nom de plume. She was the daughter of the mathematician Paul Appell.

[9]They were collected in a small brochure published by the Ecole Normale at the end of the war [ENS, 1919].

[10]L'Ecole de1914 a vengé l'Ecole de 1870, [ENS, 1919], p.3

previous generation in every domain of knowledge. They were so young that almost none had had time to start making a reputation of his own through professional achievement. As testimony of his assumed abnegation, Frédéric Gauthier, a young hellenist, who had entered the Ecole Normale Supérieure in 1909 and was killed in July 1916 in the battle of Verdun, left a melancholic comment on this time of resignation: *My studies, it is true, will remain sterile, but my ultimate actions, useful for the country, have the same value as a whole life of action* [11].
Gateaux, who died at the very beginning of the war, appears therefore to be a good representation of the lost generation of *normaliens* that I have just mentioned; he was at the same time an exception, as his very name, contrary to almost every one of his companions of misfortune, has been retained in mathematics. The way in which it has been retained, and above all the direction in which his works received their most important development (Wiener's seminal paper [Wiener, 1923]) was, at least partly, related to the war. Lévy wrote to Fréchet in 1945

> As for myself, I learned the first elements of probability during the spring of 1919 thanks to Carvallo [the director of studies at the Ecole Polytechnique] who asked me to make three conferences on that topic to the students there. Besides, in three weeks, I succeded in proving new results. And never will I claim for my work in probability a date before 1919. I can even add, and I told M.Borel so, that I had not really seen before 1929 how important were the new problems implied by the theory of denumerable probabilities. But I was prepared by functional calculus to the studies of functions with an infinite number of variables and many of my ideas in functional analysis became without effort ideas which could be applied in probability.[12]

The urgent need to renew the teaching of probability at the Ecole Polytechnique was a side effect of the war, when much probabilistic technique had been used to adjust shooting. And it is because Gateaux was dead that Lévy was in possession of his papers. Nobody can tell what would have been Lévy's career without the conjunction of these two disparate elements that his fertile mind surprisingly connected.
Therefore a focus on Gateaux allows us to shed some light on some aspects of mathematics in France before and after the Great War and to understand how such an event may have influenced their development, not only in technical aspects but also because of its terrible human cost.
The paper is divided into four parts. In the first I describe Gateaux's life before he went to Rome in 1913. Then I present the critical period in Rome with Volterra. The third part treats his departure to the army and his last days. Finally, there is a slightly more technical part which considers the work of the Gateaux and how it was recovered by Lévy and considerably extended by him so that it became a step towards the construction of an abstract integral in infinite dimensions and then of modern probability theory.

1. A PROVINCIAL IN PARIS

We do not know much about Gateaux's life before he entered the Ecole Normale. Gateaux did not belong to an important family and moreover his family unit consisted only of his parents, his younger brother Georges and himself. Neither of the brothers had direct descendants, as both

[11] Mes études, il est vrai, seront demeurées stériles, mais mes actions dernières, utiles au pays, vaudront toute une vie d'action, [Annuaire, 1918]

[12] [Barbut, 2004], p.156.

boys died during WW1. I have met a distant member of his family, namely the great-great-great-great-grandson of a great-great-great-grandfather of René Gateaux, M.Pierre Gateaux, who still lives in Vitry-le-François and most kindly offered access to the little information he has about his relative.

René Eugène Gateaux was born on 5 May 1889 in Vitry-le-François in the département of Marne, 200 km east to Paris[13]. René's father Henri, born in 1860, was a small contractor who owned a saddlery and cooperage business in the outskirts of Vitry. His mother was Marie Roblin, born in Vitry in 1864. René's family on his father side came originally from the small town of Villers-le-Sec at 20 km from Vitry, the rural nest of Gateaux's family. René's birth certificate indicates that Eugene Gateaux (Henri's father) was a proprietor and Jules Roblin (Marie's father) was a cooper; the grandparents acted as witnesses when the birth was registered at the town hall. Eugène's birth certificate indicates that he was born in Villers-le-Sec in 1821 and that his father was a carpenter. Perhaps René's grandfather came to Vitry to create his business and employed Marie's father as a cooper. As already mentioned, the couple had two children : René is the elder, the second one, Georges was born four years later in 1893. René's father died young, in 1905, aged 44, and the resulting precarious situation may have increased the boy's determination to succeed in his studies.

I have no details on René Gateaux's school career; he was a pupil in Vitry and then in Reims. The oldest handwritten document I have found is a letter to the Minister of Public Instruction on 24 February 1906 asking for permission to sit for the examination for admission to the Ecole Normale Superieure[14] (science division), although he had not reached the regular minimum age of 18.

Two things can be deduced from this document. The first is that Gateaux was a student in a Classe Préparatoire in the lycée of Reims[15]. Our second inference is that Gateaux was a brilliant student in his science classes. He should have obtained his baccalauréat in July 1904 at the age of only 15. However he was not admitted to the Ecole Normale on his first attempt in 1906, but only in October 1907 after a second year in the class of Mathématiques Spéciales as was usually the case.

What was it like to be a provincial in Paris? Jean Guéhenno, born in 1890, and admitted in 1911 in the literary section, has written some fine pages on the subject in his *Journal d'un homme de 40 ans* ([Guéhenno, 1934] - see in particular the Chapter VI, 'Intellectuel'). There he describes the Ecole Normale Superieure of the years before the Great War through the eyes of a young man from a poor provincial background (much poorer in fact than Gateaux's) and how he was dazzled by the contrast between the intellectual riches of Paris and the laborious tedium of everyday life in his his little industrial town in Brittany. We also have an obituary written in 1919 by two of Gateaux's fellow-students from the 1907 science section of the Ecole Normale, Georges Gonthiez and Maurice Janet. They wrote :

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[13]Abraham de Moivre was born there 222 years earlier, before the wars of religion forced him to leave for London where he spent all his scientific career. François Jacquier was also born there 178 years earlier. A local historian from Vitry, Gilbert Maheut, has written several short papers about his three mathematician fellow-citizens. See in particular [Maheut, 2000].

[14]After the defeat of 1870, the prestige of the Ecole Polytechnique has faded and the Ecole Normale Supérieure had become the real center of intellectual life in France at the turn of the century.

[15]The Classes Préparatoires are the special sections in the French school system that train students for the competitive examinations for entry to the 'Grandes Ecoles' such as the Ecole Polytechnique or the Ecole Normale Supérieure.

> He was one of those good comrades with whom one likes to chat. His benevolence and absolute sincerity were felt immediately; he was one of those who knew how to listen and to empathize with the other's thoughts. Maybe others were more assertive of their personality, more inclined to prove the originality of their spirit and character. It was without noise that Gateaux's personality blossomed, following the way he judged to be the best possible, and his personality unceasingly and smoothly strengthened. He had this freshness of spirit of the right nature not yet offended by life. When he arrived at the Ecole, he quietly opened his spirit to new subjects with the natural easiness and the calm of a modest, self-confident and beautiful intelligence. [. . .] He soon appeared to us as one of the best mathematicians in our group, serious-minded, and quick to focus on the essential. He liked to deal with all kinds of philosophical or general questions.[16]

After the entry at the Ecole occurred an event in the young man's life of undoubted importance since Gonthiez and Janet devote many lines to it. Gateaux became a member of the Roman Catholic Church. He joined the church *with fervour*, wrote his two fellows. Such a decision in 1908 may seem surprising: the separation laws between Church and State had been passed in 1905 and the Roman Church stood accused for its behaviour during the Dreyfus Affair. However, there was concurrently a revival of interest in Catholicism as a counterweight to triumphant positivism. Such a current was well represented at the Ecole Normale [Gugelot, 1998]. Among Gateaux's fellows was Pierre Poyet who chose a religious life and died a few months before he could make his vows as a Jesuit.

René's conversion to Catholicism, which had a profound effect on his spiritual life, created difficulties for him at the Ecole Normale. Gateaux explained in a letter to Poyet (quoted in [Bessières, 1934]) that his conversion was received badly by his fellows and some professors. Several pages are devoted to Gateaux in Béssières' biography of Poyet ([Bessières, 1934]). So far, all efforts to locate Poyet's personal papers have been fruitless, nevertheless the obituary by Gonthiez and Janet testifies to the incomprehension felt by Gateaux's fellows.

> It is indeed striking to see how [Gateaux] dealt with both his religious and secular lives. His great abilities for the intellectual work, a delicate health and the absence of any self-love may have passed for indolence. He tired rather quickly, and, if he did not feel himself in a sufficiently good mood for working, he stopped immediately and went for a walk. He rarely imposed a task on himself in advance and somehow left it to circumstances to lead him. But when he was in the right mood, he amazed us by his quickness to focus on the heart of a problem, to

[16]C'était un de ces bons camarades avec qui l'on aimait converser, dont on sentait dès l'abord la bienveillance et la sincérité parfaite, un de ceux qui savaient écouter et savaient entrer dans la pensée d'autrui. D'autres cherchaient peut-être davantage à affirmer leur personnalité, à mettre en valeur l'originalité de leur esprit et de leur caractère. Sans ces éclats et ces écarts, la personnalité de Gateaux s'est développée d'une manière harmonieuse en suivant la voie qu'elle a cru la meilleure, et elle s'est raffermie sans cesse et sans heurts. Il avait cette fraîcheur d'esprit des natures droites que la vie n'a pas encore froissées; et, arrivant à l'Ecole, il ouvrait tranquillement son esprit à de nouveaux objets avec l'aisance naturelle et le laisser-aller d'une belle intelligence sûre d'elle-même dans sa modestie. [. . .] Il nous apparut bientôt comme un des tous premiers mathématiciens de la promotion, réfléchi, sérieux, prompt à découvrir l'essentiel, aimant à se porter vers les questions d'intérêt général et philosophique. ([Annuaire, 1918], pp.136 to 140)

> organize an Agrégation lecture[17] around a central, though sometimes a little too elevated, idea. One could hear him jump with exclamations as he went along his discoveries[18][19].

In 1910, Gateaux passed the Agrégation of mathematical sciences where he obtained the 11th rank out of 16. This was not a very good rank so that it left him no possibility of obtaining a grant to devote himself entirely to research as had been the case for Joseph Pérès, for instance (on which I shall comment later). On 8 July 1912, a ministerial decree appointed Gateaux as Professor of Mathematics at the Lycée of Bar-le-Duc, the principal town of the département of Meuse, 250 km east from Paris, and not very distant from his native town.
Before taking up this position, Gateaux had to fulfill his military obligations. From March 1905 ([Journal Officiel, 1905]), a new law replaced the July 1889 regulation for the organization of the army. The period of active military service had been reduced to 2 years, but conscription became in theory, absolutely universal.
Gateaux was particularly concerned by article 23 of the 1905 law. It stipulated that the young men who entered educational institutions such as the Ecole Normale Supérieure could, at their choice, fulfill the first of their two years of military service in the ranks before their admission to these institutions or after their exit. In the latter case (generally chosen by the students in order not to interrupt their schooling, they were supposed to receive a military instruction preparing them to the grade of second lieutenant of the reserve. Then, after the end of their studies, they must serve one year in the ranks and then serve for one year as second lieutenant (in the reserve or in the active army). Gateaux chose the latter option ([Gateaux, 1922b]). The students were asked to sign a voluntary five year engagement before entering the school (that is to say their 3 years of Ecole and two years of active military service), which Gateaux did in Vitry in October 1907. In October 1910, Gateaux joined the 94th Infantry regiment where he was a private. In February 1911 he was promoted *caporal*[20], and finally was declared second lieutenant in the reserve in September 1911. He had to follow some special training for officers; the comments made by his superiors on the military file indicate that the supposed military training at the Ecole Normale had been more virtual than real. On the special pages devoted to his superior's appraisal, one reads

[17]That is to say a mock lecture of the type that candidates to the Agrégation competition - the degree to obtain a position as a secondary school teacher - has to present to the board of examiners.

[18]Il est en effet absolument frappant de mettre en parallèle les méthodes employées par Gateaux dans sa vie religieuse et dans sa vie extérieure. Ses grandes facilités dans le travail intellectuel, une santé assez délicate et l'absence de tout amour-propre lui avaient composé des habitudes qui semblaient tenir de la mollesse Il se fatiguait assez vite, et, s'il ne se sentait pas bien en forme pour entreprendre un travail, il le lâchait sans tarder et allait se promener; il s'imposait rarement une tâche à l'avance et se laissait un peu guider par les circonstances. Mais , lorsqu'il était dispos, il nous émerveillait par sa rapidité à saisir le nœud d'un problème, à grouper autour d'une idée centrale, parfois un peu trop élevée, une leçon d'agrégation. On l'entendait bondir d'exclamations en exclamations au fur et à mesure de ses découvertes.

[19]Later, Gonthiez and Janet recalled an amazing sentence written by Gateaux in his diary (which I have not been able to locate) that an *ecclesiastic who followed him closely* communicated to them : *I asked God to make a saint out of me... Maybe I shall need to resign from my profession and to follow Jesus and preach. I do not know how this would be done. Maybe God would also ask me to stay in the University. I shall know that later.* [J'ai demandé à Dieu la grâce de faire de moi un Saint... Il faudra peut-être quitter ma profession et suivre Jésus, me livrer à la prédication. Je ne sais sous quelle forme cela se réalisera. Peut-être aussi Dieu me demandera-t-il de rester dans l'Université. Je le saurai en avançant.] Bessières' book [Bessières, 1934] provides a surprising picture of the mystic atmosphere present at the Ecole Normale around Poyet.

[20]A title corresponding to a kind of private first class.

: *1st Semester 1912 : M.Gateaux strives to acquire the aptitude for command and for the duties of a second lieutenant. He has very good spirit, is very intelligent, zealous, and conscientious, but he was hardly prepared for his rank*[21]. The second semester 1912 (which ended in fact in September 1912) seems however to have been better [22] and the final comment has a strange resonance with what happened two years later. Gateaux's superior mentioned that he was *able to lead a machine-gun section*[23].
In October 1912, Gateaux, freed from the active army, began his lectures at the Lycée of Bar-le-Duc. Gateaux's (very thin) personel file[24] contains a personal identification form and a decree of the Minister of Public instruction on 2 October 1913 granting him one-year's leave with an allocation of 100 francs for that year, as well as a handwritten document showing that he had obtained a David Weill grant for an amount of 3000 francs.

2. The Roman stage

Gateaux had indeed begun to work on a thesis with themes closely related to functional analysis *à la Hadamard*. I have found no precise information about how Gateaux chose this subject for his research, but it is plausible that he was advised to do so by Hadamard himself. In 1912, Hadamard had just delivered a series of lectures on functionals at the Collège de France and had entered the Academy of Science in the same year. Paul Lévy had moreover defended his own brilliant thesis on similar questions in 1911. As well, a young French normalien of the year before Gateaux, Joseph Pérès, had in 1912-1913 benefited from a David-Weill grant offered for a one-year stay in Rome with Volterra. Volterra himself, invited by Borel and Hadamard, had come to Paris for a series of lectures on functional analysis, edited by Pérès and published in 1913 ([Volterra, 1913]). These were thus good reasons for Gateaux to be attracted by this new and little explored domain. For a young doctoral student the natural people to be in contact with were Hadamard in Paris and Volterra in Rome[25]. Pérès's example encouraged Gateaux to go to Rome. Some years later, when Hadamard wrote a report commending the posthumous attribution of the Francœur prize to Gateaux, he mentioned that the young man had been *one of those who, inaugurating a tradition*

[21]1er semestre 1912 : M.Gateaux s'efforce d'acquérir l'aptitude au commandement et à ses fonctions de sous-lieutenant. Il a très bon esprit, est très intelligent, zélé et consciencieux, mais il était bien peu préparé à son grade.

[22]It is written that Gateaux *has made much progress. Very intelligent, very conscientious, and very good natured, willing to do the best. He has become a good section leader, capable of useful tasks in case of mobilization. He has followed a period of instruction for shooting and obtained very good marks*. [A beaucoup progressé, fort intelligent, très consciencieux et de bons sentiments, soucieux d'arriver à bien faire. Il est devenu un bon Chef de section, capable de rendre de bons services à la mobilisation. A fait un stage à l'Ecole de tir et y a obtenu de très bonnes notes.]

[23]Est apte à conduire une section de mitrailleuses.

[24]Quite unfortunately, the archives of the Lycée in Bar have rotted in the cellar of the institution and the few surviving documents have recently been transfered to the Archives of the département of Meuse. Among them, the register of salaries on which one can follow the evolution of Gateaux's salary since his appointment in October 1912.

[25]On Hadamard, a star of the French mathematical stage of the time, the reader can refer to the book [Mazya and Shaposhnikova, 1998]. Two biographies of Vito Volterra have recently been published ([Goodstein, 2007], [Guerraggio and Paoloni, 2008]), and the reader can also find information in the annotated edition of the correspondence between Volterra and his French colleagues during WW1 ([Mazliak and Tazzioli, 2009]).

that could not be overestimated, went to Rome to become familiar with M.Volterra's methods and theories[26].

On the occasion of the centennial of Volterra's birth, in 1960, a volume was edited by the Accademia dei Lincei in Rome in which Giulio Krall devoted several pages to Volterra's research on the phenomenon of *hysterisis*, the 'memory of materials', which describes the dependence on time of the state of deformation of certain materials. To model such a situation, Volterra was led to consider *functions of lines* (funzione di linea), later called *functionals* (fonctionnelle) by Hadamard and his followers, which is to say a function of a real function representing the state of the material, and to study the equations they must satisfy. These equations happen to be an infinite-dimensional generalization of partial differential equations. As Krall mentions[27], *from mechanics to electromagnetism, the step was small*, and Volterra's model was applied to different physical situations, such as electromagnetism or sound produced by vibrating bars[28]. In 1904, the King made Volterra a Senator of the Kingdom, mostly honorary, but giving the recipient some influence through his proximity with the men of power.

Such a combination of science and politics appealed to Borel who had a deep friendship with Volterra[29]. Borel had a part in Gateaux's decision to go to Rome, at least as an intermediary between the young man and Volterra. We indeed find a first indication of this Roman project in their correspondence. Borel wrote to Volterra

> Another young man, who is also my former student, M.Gateaux and presently teacher at Bar-le-Duc lycée, informed me of his intention to solicit a study grant to continue his research. I recommended he apply for a David Weill grant and go to Rome, if you would welcome him. I enclose his letter where he mentions that he asked for the grant, and presents what he intends to do. He is less advanced than Pérès was, as part of his time is devoted to teaching, while Pérès had a study grant. I am also inclined to think that he may be less distinguished than Pérès, but I think nevertheless that he will derive benefit from the grant he would obtain and I have the intention to support his request[30].

Borel then asked Volterra to write a short letter of support for this project to Liard, the Vice-Rector of the Paris Academy, and also mentioned that he lent the two books published by Volterra on the functions of lines to Gateaux (more precisely, the book [Volterra, 1912] and the proofs of

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[26]Il fut un de ceux qui, inaugurant une tradition à laquelle nous ne saurions trop applaudir, allèrent à Rome se former aux méthodes et aux théories de M.Volterra.([Hadamard, 1916]). On the development of student exchanges between Paris and Rome in these years, see [Mazliak, 2012].

[27][Krall, 1961], p.17.

[28]Volterra himself was involved in this subject through an important collaboration with Arthur Gordon Webster from Clark University in USA. See the interesting webpage **http://physics.clarku.edu/history/history.html#webster**.

[29]On the beginning of the relationship between Borel and Volterra, see [Mazliak, 2012].

[30]Un autre jeune homme qui est aussi mon ancien élève, M.Gateaux, actuellement professeur au Lycée de Bar-le-Duc, m'a parlé récmment de ses intentions de demander une bourse d'étude en vue de recherches qui se rattachent à vos travaux. Je lui ai conseillé de demander comme Pérès une bourse David Weill et d'aller à Rome, si vous voulez bien l'accueillir. Je vous communique ci-inclus la lettre dans laquelle il me fait part qu'il a demandé la bourse, et m'indique ses intentions. Il est moins avancé que n'était Pérès, car il a la charge d'un l'enseignement, tandis que Pérès avait une bourse d'étude. Je serais aussi porté à croire qu'il est peut-être moins distingué que Pérès . Mais je le crois néanmoins capable d'utiliser avec fruit la bourse qui lui serait accordée et j'ai l'intention d'appuyer sa demande. (Borel to Volterra, 18 April 1913)

[Volterra, 1913]). On 30 June 1913, Borel communicated the good news to Volterra : a David Weill grant had been awarded to Gateaux for the year 1913-1914.
Gateaux's aforementioned letter to Borel[31] was in Volterra's archives, and consequently we know precisely what his mathematical aims were when he went to Rome. Gateaux considered two main points of interest for his future research. The first one is classified as *Fonctionnelles analytiques* (Analytical functionals), and is devoted to the extension of the classical results on analytical functions : the Weierstrass expansion, the equivalence between analyticity and holomorphy and the Cauchy formula.
Gateaux presented his program as follows :
Gateaux started from the definition Fréchet had proposed in 1910 for an analytical functional ([Fréchet, 1910]) based on a generalization of a Taylor expansion. A functional[32] U is homogeneous with order n if for any $p \geq 1$ and any given continuous functions $f_1, \dots, f_p$ over $[a, b]$, the function defined on $\mathbb{R}^p$ by

$$(\lambda_1, \dots, \lambda_p) \mapsto U(\lambda_1 f_1 + \dots + \lambda_p f_p)$$

is an homogeneous polynomial of degree less than n[33]. Now, a functional U is analytical if it can be written as

$$U(f) = \sum_{n=0}^{\infty} U_n(f)$$

where U_n are homogeneous functionals of order n ([Fréchet, 1910], p.214).
Gateaux first proposed a more careful study of the expression of $U_n(f)$. Then, he intended to obtain the equivalence between the analyticity of the functional U and its complex differentiability (holomorphy) and to deduce a definition of analyticity by a Cauchy formula. For that purpose, as he wrote, one needs a definition of the integral of a real continuous functional over a real functional field. This may be the first appearance of questions around infinite dimensional integration. In this programmatic letter, Gateaux suggested the way he wanted to proceed, inspired by Riemann integration.

> Let us restrict ourselves to the definition of the integral of U in the field of the functions $0 \leq f \leq 1$. Let us divide the interval (0,1) into n intervals. [...] Consider next the function f in any of the partial intervals as equal to the numbers $f_1, \dots, f_n$ which are between 0 and 1. $U(f)$ is a function of the n variables $f_1 \dots f_n : U_n(f_1, \dots, f_n)$. Let us consider the expression
>
> $$I_n = \int_0^1 \int_0^1 \dots \int_0^1 U_n(f_1, \dots, f_n) df_1 \dots df_n.$$
>
> Suppose that n increases to infinity, each interval converging to 0, and that I_n tends to a limit I independent of the chosen divisions. We shall say that I is the integral of U over the field $0 \leq f \leq 1$.

[31]Dated from Bar-le-Duc, 12 April 1913, reproduced and translated in Appendix 1.
[32]Throughout the paper, the functionals considered are always defined on the set of real functions over a given interval $[a, b]$.
[33]Fréchet's definition is in fact given in a different way by means of a property inspired by a characterization he had proved for real polynomials ([Fréchet, 1910], p.204); however, he proves (p.205) that both properties are equivalent.

Gateaux's intention was to study whether the limit I exists for any continuous functional U, or if an extra hypothesis was necessary. In the last paragraph, Gateaux mentioned the possible applications of this integration of functionals, such as the residue theorem. All the applications he mentioned belong in addition to the theory of functions of line. There is no hint of a possible connection with potential theory. This does not appear in the papers published by Gateaux. As it is a central theme of Gateaux's posthumous texts, it is plausible that he became conscious of the connection only during his stay in Rome - perhaps under Volterra's influence.

On 28 August 1913, Gateaux wrote directly to Volterra for the first time, informing him of his arrival in October, and also mentioning that he had already obtained several results for the thesis in Functional Analysis on which he was working on. Gateaux may have enclosed a copy of his first note to the Comptes-Rendus ([Gateaux, 1913a]), published on 4 August 1913 and containing the beginning of his proposed program. The note is in fact rather limited to an exposition of results and does not contain any proof, apart from a sketchy construction of uniform approximation of a continuous functional $U(z)$ by functionals of order n[34]

About Gateaux's stay in Rome, I do not have many details. An interesting document, found in the Paris Academy, is the draft of a report written by Gateaux at the end of his stay for the David Weill foundation[35]. He mentioned there that he had arrived in Rome in the last days of October, and that he followed two of Volterra's courses in Rome (one in Mathematical Physics, the other about application of functional calculus to Mechanics). Gateaux seems to have worked quite actively in Rome. A first note to the Accademia dei Lincei ([Gateaux, 1913b]) where he extended the results of his previous note to the Paris Academy was published in December 1913. On a postcard sent by Borel to Volterra on 1 January 1914, Borel mentioned how he was glad to learn that Volterra was satisfied with Gateaux. The young man published three more notes during his stay ([Gateaux, 1914a],[Gateaux, 1914b],[Gateaux, 1914c]), but also began to write more detailed articles - found after the war amongst his papers[36].

On 14 February 1914, Gateaux made a presentation to Volterra's seminar[37], in which he mainly dealt with the notion of the functional differentiation. He recalled that Volterra introduced this notion to study problems including hereditary phenomena, but also that it was used by others (Hadamard and Paul Lévy) to study some problems of mathematical physics - such as the equilibrium problem of fitted elastic plates - through the resolution of equations with functional derivatives.

Gateaux came back to France at the beginning of the summer, in June 1914. He expected to go back soon to Rome as he was almost certain, as Borel had written to Volterra[38], to obtain

[34]It is not worthy to dwell upon this technical result here, which had already been obtained by Fréchet previously ([Fréchet, 1910], p.197) in a slightly more intricate way. Let me only observe that Gateaux's elementary technique involves the replacement of the function z by a linear function over each subdivision $[\frac{i}{n}, \frac{i+1}{n}]$ of the interval [0,1]. A final perfecting of Gateaux's proof is presented by Lévy in [Lévy, 1922], pp.105-107.

[35]A very touching aspect of the report written by Gateaux for the David-Weill foundation can be found in the pages where he described the non-mathematical aspects of his journey. Gateaux mentioned how he regretted that Italy and the Italian language were so ignored in France, when, on the contrary, France and French were widely known within Italian society.

[36]Lévy (in [Gateaux, 1919b], p.70) mentioned that in one case, two versions of the same paper were found, both dated March 1914.

[37]His lecture notes were found among his papers.

[38]Borel to Volterra. 3 April 1914

the Commercy grant he had applied for. Gateaux soon wrote that the grant was accepted[39]. In the same letter, he mentioned that he had completed a first version of a note on functionals requested by Volterra for appending it to the German translation of his lectures on functions of lines ([Volterra, 1913]). During this month, he had also met the Proviseur of the Lycée in Bar-le-Duc on July 20th, as the man sadly observed in a letter after Gateaux's death[40].

3. IN THE STORM

A serious danger of war had in fact been revealed only very late in July 1914 in public opinion, and the French mostly received the mobilization announcement on August 2nd with stupor. As the majority, Gateaux has been caught napping by the beginning of the war. He was mobilized in the reserve as lieutenant of the 269th Infantry regiment, member of the 70th infantry division. The diaries of the units engaged in the war[41] permit us to follow Gateaux's part in the campaign in a very precise way. He was appointed on August 6th as the head of the 2nd machine-gun sections of the 6th Brigade when the unit was formed in Domgermain in suburbs of the city of Toul[42]. The regiment paused beyond Nancy the next day and was supposed to go further East but the German army's fire power stopped it brutally a few days later near Buissoncourt, 15 kilometers east of Nancy. At the end of August, the main task of the 70th infantry division was to defend Nancy's south-east sector.

It is hardly recognized today how horrific the first few weeks of the war were on the French side. August 1914 was the worst month of the whole war in terms of casualties, and some of the figures defy belief. On 22 August 1914, for example, the most bloody day of the whole war for the French, 27,000 were killed in the French ranks ([Becker, 2004]). The appallingly high number of casualties was due to an alliance between the vulnerability of the French uniform (with the famous *garance* (red) trousers up to 1915...), the self-confidence of the headquarters who had little consideration for their men's lives, and the clear inadequacy of many leaders in the field. Prochasson[43] advances two hypotheses to explain why the casualties among the Grandes Ecoles' students (Ecole Normale Supérieure in particular) were so dramatic. As they were often subordinate officers, the young students were the first killed as their rank placed them in the front of their section. But also, they were sometimes moved by a kind of stronger patriotic feeling that may have driven them to a heroism beyond their simple duty[44]. This is evident in Marbo's testimony about her adopted son Fernand, who explained to her that, as a socialist involved in the fight for the understanding between peoples and peace, he wanted to *be sent on the first line in order to prove that he was as brave as anyone else*[45], and added that *those who would survive will have the right for speaking loudly in front of the shirkers*[46].

[39]Gateaux to Volterra, 14 July 1914

[40]Postcard dated from 7 December 1914, reproduced in Appendix 3.

[41]They were put on-line by the French Ministry of Defense http://www.memoiredeshommes.sga.defense.gouv.fr .

[42]Gateaux used headed notepaper from the *Hotel & café de l'Europe* in Toul for his last letter to Volterra on August 25th, reproduced in Appendix 2.

[43][Prochasson, 2004], pp.672-673.

[44]Prochasson mentions the famous example of Charles Péguy, and the less well-known one of the anthropologist Robert Hertz who unceasingly asked his superiors for a more exposed position, and was killed in April 1915.

[45]être envoyé en première ligne afin de prouver qu' [il était] aussi courageux que n'importe qui

[46]Ceux qui survivront auront le droit de parler haut devant les embusqués. ([Marbo, 1967], p.166)

Gateaux's last letter to Volterra is dated August 25th, and is reproduced and translated in Appendix 2. Gateaux alluded there to the ambiguous situation of Italy. Though officially allied to the Central Empires, the country had carefully proclaimed its neutrality, an interesting point described at length in [Rusconi, 2005]. Senator Volterra immediately sided with France and Great Britain and wrote passionate letters to his French colleagues as early as the beginning of August to express the hope that Italy would join them[47]. On 24 October 1914, in a letter to Borel, he asked for news

> from M.Gateaux, M.Pérès, M.Boutroux and M.Paul Lévy and other young French friends. I have received a letter from M. Gateaux from the battlefield and then no other. And this is why I am very worried about his fate and that of the others[48].

Borel answered Volterra's letter on November 4, telling him that Pérès and Boutroux were discharged and that he did not know where Gateaux was[49].As we have seen, Gateaux was in Lorraine at the end of August. The French army went steadily backward, and was closer and closer to being crushed between the two wings of the German army (one coming from the north through Belgium, the other from the east through Lorraine and Champagne). Then occured the unexpected *miracle* of the Battle of the Marne (6-13 September 1914) which suddenly stopped the German advance, rendering the Schlieffen Plan a failure. Vitry-le-François had been occupied by the Germans during the night of the 5th of September, but they were compelled to leave and to withdraw towards the East on September 11th[50]. From September 13th, the French went again slowly towards the East, chasing after the retreating Germans.

At the end of September, the French and British and the German headquarters became aware of the impossibility of any further decisive motion on the front line running from the Aisne to Switzerland; each realized that the only hope was to bypass their enemy in the zone between the Aisne and the sea which was still free of soldiers.

General Joffre decided to withdraw from the Eastern part of the front (precisely where Gateaux was) a large number of divisions and to send them *by railway* to places in Picardie, then in Artois and finally to Flanders to try to outrun the Germans. The so-called *race for the sea* lasted two months and was very bloody.

The 70th division was transported between September 28th and October 2nd from Nancy to Lens, a distance of almost 500 km[51]. Gateaux's division received the order to defend the East of Arras.

[47]See [Mazliak and Tazzioli, 2009] where Volterra's attitude is thoroughly studied.

[48]M.Gateaux, M.Pérès, M.Boutroux, M.Paul Lévy et d'autres jeunes amis français. (...) J'avais reçu une lettre de M.Gateaux du champ de bataille et ensuite je n'en ai reçu pas d'autre c'est pourquoi je suis très inquiet sur son compte ainsi que sur les autres.

[49]The tone of this letter was slightly less confident than the previous ones. This was the moment when the enormous losses of the first weeks began to filter through. Borel wrote that at the Ecole Normale, several young men with a bright scientific future had already disappeared and that the responsibility of *those who wanted this war* was really terrible.

[50]A vivid account of this moment was written after the war by a witness ([Nebout, 1922]). Though Gonthiez and Janet wrote in [Annuaire, 1918] that they could *easily imagine all the pain he [Gateaux] would have felt when he learned that the enemy had taken the city of Vitry-le-François where his poor mother had remained*, it is not clear whether Gateaux had learnt the fact at all, due to the general confusion. I refer to [Becker, 2004] or to several articles of [Encyclopédie, 2004] for the description of this phase of the war.

[51]According to the diary of the 269th Infantry regiment, the order to board the trains, received on September 28th, was carried out the next day. With an impressive organizational efficiency, the trains followed a circuitous route to join Artois : Troyes, Versailles, Rouen before stopping at Saint-Pol sur Ternoise on October 1.

On October 3rd, Gateaux's regiment was in Rouvroy, a small village, 10km South-East from Lens and Gateaux was killed at one o'clock in the morning, while trying to prevent the Germans from entering the village. In the confusion of the bloodshed, the corpses were not identified before being collected and hastily buried in improvised cemeteries. Gateaux's body was buried near St Anne Chapel in Rouvroy, a simple cross without inscription marking the place[52].
René's death was officially established only on 28 December 1915[53]. But it is only long after, on 8 December 1921, that Gateaux's corpse was exhumed and formally identified, and finally transported to the necropolis of the military cemetery of the Bietz-Neuville St Vaast[54]. The last document of the military dossier is a letter from the Minister of War, dated 22 June 1923, informing the mayor of Vitry-le-François that the Lieutenant René-Eugène Gateaux had officially been declared *Dead for France*.
The detailed chronology of how the academic world learned of Gateaux's death is not entirely clear. As already mentioned, the Principal of Bar-le-Duc Lycée wrote the postcard reproduced in Appendix 3 in December 1914, but it was clearly an answer to a letter he had received[55].
Only on December 10th did Borel write to Volterra about Gateaux's death.

> The success will unfortunately cost irreplaceable losses ; among the sad news I have recently heard, one that caused me most grief is Gateaux's death. The conditions in which it was announced to us leave unfortunately the tiniest hope of a mistake. I want though to hope that of the dozens of pupils of the Ecole Normale considered as lost, there will be at least one or two who will come back at the end of the war ([Mazliak and Tazzioli, 2009], p.47).

Volterra sadly answered some days later

> Gateaux was very talented and I am sure that he had a great future. He was developing his ideas in a somewhat slow but always precise way. Last year, he had worked a lot and I did not doubt that all the material for his thesis was ready. How many young lives have been the victims of this war ! It is horrible to think ([Mazliak and Tazzioli, 2009], p.48)

The same day a telegram was sent to the Ecole Normale by Volterra in the name of the Mathematical seminar in Rome.
As early as August 1915, Hadamard took the necessary steps to obtain the award of one of the Paris Academy's prizes for Gateaux. In a letter dated 5 August 1915 (and probably addressed to Picard as Perpetual Secretary), Hadamard mentioned : Gateaux *has left very advanced research on functional calculus (his thesis was composed to a great extent, and partly published in notes*

[52]According to the army file, René's mother was informed on October 4 that her son was reported missing. On March 16th 1916, her other son and only remaining child, René's brother Georges was killed in the Mort-Homme before Verdun. Much later, René's mother passed away on 24 February 1941 in Vitry-le-François, some months after having seen her city annihilated by the German invasion.

[53]This was done based on evidence given by Henri-Auguste Munier-Pugin, warrant officer and Albert Garoche, sergeant in the 269th Infrantry regiment.

[54]Gateaux's grave is number 76 at Bietz-Neuville. Gateaux's mother was informed of the fact on 5 January 1922.

[55]This postcard is however a decisive link between Hadamard and the papers left by Gateaux. It was probably addressed to Hadamard or Borel, though I found it by chance in the huge archive of Fréchet material in the Paris Academy of Science. Another possibility is that the letter was addressed to Fréchet who happened to know the Proviseur as well as Gateaux well enough to have this exchange. If this hypothesis is true, it may be Fréchet who recovered Gateaux's papers and transmitted them to Hadamard. We shall see a point below that corroborates this version.

to the Academy), research for which M.Volterra and myself have a great regard[56]. At the meeting of 18 December 1916, the Francœur prize was awarded to Gateaux ([Hadamard, 1916], pp.791-792). It is interesting to read in Hadamard's short report the following section

> [Gateaux] was following a much more audacious way, which promissed to be very fruitful, by extending the notion of integration to the functional domain. Nobody could predict the development and the range this new series of research would attain. This is what has been interrupted by events[57].

It is plausible that Hadamard had only superficially looked at Gateaux's papers, since he himself was caught in the storm of events, losing his two sons during the summer of 1916. Nevertheless, he did at least notice that one major interest in the last period of the Gateaux's work was integration over the space of functionals. As we shall see, this was precisely why he spoke to Lévy about Gateaux.

4. The mathematical destiny

4.1. **Lévy's interest in infinite dimensional integration.**

> In January 1918, I was lying on a bed in a hospital, when I suddenly thought again of functional analysis. In my early work, I had never thought of extending the notion of an integral to spaces with infinite dimensions. It suddenly appeared to me that it was possible to attack this problem starting with the notion of mean in a sphere of the space of square summable functions. Such a function can be approximated by a step function, the number n of its distinct values growing constantly. The desired mean may then be defined as the limit of the mean in a sphere of the n-dimensional space. Obviously, this limit may not exist; but in practice, it does often exist. ([Lévy, 1970], p.58)

Thus Lévy described how he became interested in infinite dimensional integration. It is not easy to decide whether this happened as suddenly as he wrote, just following the train of his thoughts. Regardless, it is sometimes forgotten today that Lévy, before becoming one of the major specialists in Probability theory of 20th Century, had been a brilliant expert in functional analysis[58]. As we shall see, it is a remarkable fact that his studies in functional analysis led him rather naturally to probabilistic formulations of problems. At the end of 1918, the Paris Academy of Sciences, following Hadamard's proposal, decided to call upon Lévy for the *Cours Peccot* in

[56][Gateaux] laisse sur le calcul fonctionnel des recherches fort avancées (sa thèse était en grande partie composée, et représentée par des notes présentées à l'Académie), recherches auxquelles M.Volterra, comme moi-même, attache un grand prix.

[57][Gateaux] allait s'engager dans une voie beaucoup plus audacieuse, et qui promettait d'être d'être des plus fécondes, en étendant au domaine fonctionnel la notion d'intégrale. Nul ne peut prévoir le développement et la portée qui auraient pu être réservés à cette nouvelle série de recherches. C'est elle qui a été interrompue par les événements.

[58]On that topic, see in particular [Barbut, 2004], pp.44-54.

1919[59]. Lévy's book *Leçons d'Analyse Fonctionnelle* ([Lévy, 1922]), on which I shall comment later, is based on these Peccot lectures.
The first document in which the question is explicitly mentioned is a letter to Volterra written in the early days of 1919

> As I was recently interested in the question of the extension of the integral to functional space, I spoke about the fact to M.Hadamard who mentioned the existence of R.Gateaux's note on the theme. But he could not give me the exact reference and I cannot find it. [...] Though I am still mobilized, I work on lectures I hope to give at the Collège de France on the functions of lines and equations with functional derivatives, and on this occasion I would like to develop several chapters of the theory. [...] I think that the generalization of the Dirichlet problem must present greater difficulties. Up to now, I was not able to extend your results on functions of the first degree and your extension of Green's formula. This is precisely due to the fact that I do not possess a convenient expression for the integral[60].[61]

As can be seen from this quotation, Lévy's views on infinite dimensional integration were related to his studies in potential theory. The central problem of the classical mathematical potential theory is to find a harmonic function U in a domain R with given values on the border S (Dirichlet problem) or given values of the normal derivatives on S (Neumann problem). In 1906, Hadamard ([Hadamard, 1906]) proposed to make use of variational techniques from Volterra's theory of functions of lines in order to study more general forms of these problems, for instance when the border is moving with time, and in particular to find Green functions used in the integral representation of the solutions. These problems would make up Levy's thesis, defended in 1911. As Lévy wrote to Volterra, to study these questions in infinite dimensional functional spaces, one needs to be able to integrate over these spaces. Volterra was not the only person Lévy had contacted. he wrote to Fréchet on the same topic at the very end of the year 1918[62]. Fréchet had indeed proposed a theory of integration over abstract spaces in 1915, usually considered as the first attempt to define a general integral[63].

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[59] The Cours Peccot was (and still is) a series of lectures in mathematics given at the Collège de France and financed by the Peccot Foundation. It is a way to promote innovation in research by offering financial support and an audience to a young mathematician. Borel had been the first lecturer in 1900, followed by Lebesgue. In Lévy's time, the age of the lecturer was meant to be less than thirty. However, the losses of the war had been so heavy among young men that the choice of the thirty-three year old Lévy was reasonable. It is also plausible to think that Gateaux would have been a natural Peccot lecturer had he survived the war. As Lévy's appointment is almost concomitant with Hadamard asking to take care of Gateaux's papers, it is possible that there is a connection between the two events.

[60] M'étant occupé récemment de la question de l'extension de la notion d'intégrale multiple à l'espace fonctionnel, j'en ai parlé à M.Hadamard qui m'a signalé l'existence d'une note de R.Gateaux sur ce sujet. Mais il n'a pas pu m'en donner la référence exacte et je ne puis réussir à la trouver. [...] Quoiqu'encore mobilisé, je travaille à préparer un cours que j'espère professer au Collège de France sur les fonctions de lignes et les équations aux dérivées fonctionnelles et à cette occasion, je voudrais développer davantage certains chapitres de la théorie. [...] Je crois que la généralisation du problème de Dirichlet doit présenter plus de difficultés. Je n'ai pu jusqu'ici profiter pour le cas général de vos travaux sur les fonctions du premier degré et l'extension de la formule de Green. Ceci tient précisément à ce que je n'ai pas encore mis la notion d'intégrale multiple sous une forme commode pour ce but.

[61] Lévy to Volterra, 3 January 1919

[62] See [Barbut, 2004], p.69.

[63] See for instance [Kolmogorov, 1933].

On 6 January 1919, Lévy wrote to Fréchet

> About Gateaux's papers, I learned precisely yesterday that M.Hadamard had put them in security at the Ecole Normale during the war and had just taken them back. Nothing is therefore yet published.[64]

From this, I may infer that Fréchet mentioned Gateaux's papers to Lévy, probably because he had an idea of what they contained. This could also be a hint that the papers arrived to Hadamard during the war via Fréchet, and that Fréchet was the addressee of the postcard from the Principal of Bar-le-Duc.
On January 12, Lévy sent another letter to Volterra :

> M.Hadamard has just found several of Gateaux's unpublished papers at the Ecole Normale. I have not seen them yet but maybe I'll find what I am looking for in them[65].

Volterra answered on January 15, writing that none of Gateaux's publications concerned integration. He nevertheless added

> Before he left Rome, we had chatted about his general ideas on the subject, but he did not publish anything. I suppose that in the manuscripts he had left, one may probably find some notes dealing with the problem. I am happy that they are not lost and that you have them in hand. The question is very interesting[66].

As already mentioned, Hadamard entrusted Lévy with the posthumous edition of Gateaux's papers. In February 1919, Lévy began to describe to Fréchet the precise content of what he had found in Gateaux's papers to Fréchet.

4.2. **Gateaux's integration of functionals.** Integration over infinite dimensional spaces was certainly the most important subject considered by Gateaux. This may be read in Hadamard's comment that follows :

> The fact that he chose functional calculus reveals a broad mind, scornful of small problems or of the easy application of known methods. But the event proved that Gateaux was able to consider such a study under its wider and more suggestive aspect. And it is what he indeed did, with integration over the functional field, to speak only about this example, the most important, that represents an entirely new path and the very great potential offered by the theory[67].

[64][Barbut, 2004], Lettre 2.

[65]M.Hadamard vient de trouver plusieurs mémoires non publiés de Gateaux à l'Ecole Normale. Je ne les ai pas encore vus mais peut-être y trouverais-je ce que j'y recherche.

[66]Nous avons causé avant son départ de Rome des idées générales sur ce sujet mais il n'a rien publié là-dessus. Je pense que dans les notes manuscrites qu'il a laissées, on pourra bien probablement trouver quelques notes sur ce sujet. Je suis heureux qu'elles ne soient pas perdues et qu'elles se trouvent dans vos mains. La question est très intéressante.

[67]Le fait qu'il ait choisi le calcul fonctionnel révélait un esprit aux vues larges, dédaigneux du petit problème ou de l'application facile de méthodes connues. Mais le fait prouva que Gateaux était capable de considérer une telle étude sous son aspect le plus large et le plus suggestif. Et c'est effectivement ce qu'il fit, avec l'intégration sur le champ fonctionnel, pour ne mentionner que cet exemple, le plus important, qui représente une voie entièrement nouvelles et de très grandes perspectives pour la théorie. ([Annuaire, 1918], p.138)

Gateaux's views on integration are the subject of the first paper edited by Lévy in 1919 ([Gateaux, 1919a]). Lévy completed this presentation (and considerably extended it) in Part III of [Lévy, 1922] (chapter II, p.274).

As said before when I commented on Gateaux's letter to Volterra exposing his research program, Gateaux's interest in infinite dimensional integration originated in the extension of Cauchy's formula and his first idea was to use a Riemann-type approach.

Gateaux considered the ball[68] $\int_0^1 x(\alpha)^2 d\alpha \leq R^2$ of the Hilbert space of square integrable functions over [0,1][69]. He said a function x to be *simple of order* n if it assumes constant values $x_1, x_2, \dots, x_n$ over each subinterval $[0, \frac{1}{n}], \dots, [\frac{n-1}{n}, 1]$. It is therefore seen that for a simple function x to belong to the ball, one must have $x_1^2 + x_2^2 + \cdots + x_n^2 \leq nR^2$. The set of simple functions of order n belonging to the ball is called the n-th section of the ball. It is a ball in $\mathbb{R}^n$ centered at 0 with radius $\sqrt{n}R$.

As the volume V_n of a ball with radius $\sqrt{n}R$ in dimension n is asymptotically equivalent to $\frac{(2\pi e)^{n/2}}{\sqrt{n\pi}} R^n$ ([Lévy, 1922], p.265), it tends towards zero or infinity as n does, depending on the value of R. This fact constitutes the central problem for the definition of the integral : in functional space, a subset has generally a volume equal to zero or infinity, and this forbids the direct extension of the Riemann integral through an approximating step-function sequence.

Gateaux seems to have been the first to propose a natural way to bypass the problem by considering the integral as an asymptotic mean value. Consider a functional U defined and continuous on the ball $\int_0^1 x(\alpha)^2 d\alpha \leq R^2$. Its restriction U_n to the n-th section can be considered as a continuous function of the n variables $x_1, x_2, \dots, x_n$ and therefore it admits a mean value

$$\mu_n = \frac{\int_{x_1^2+x_2^2+\cdots+x_n^2 \leq nR^2} U_n(x_1, \dots, x_n) dx_1 \dots dx_n}{V_n}.$$

In some circumstances, the sequence (μ_n) admits a limit which is called the mean value of U over the ball of the functional space. Gateaux's main achievement in [Gateaux, 1919a] was to obtain the value of the mean for important types of functionals.

He began by considering functionals of the type $U : x \mapsto f[x(\alpha_1)]$ where x is a point of the functional space (a 'line'), f a continuous real function and α_1 a fixed point in [0,1]. As α_1 is fixed, $x(\alpha_1)$ is one of the coordinates when x is taken in the n-th section[70].

Therefore the ($n-1$ dimensional) volume of the intersection of the ball of radius R with the plane $x(\alpha_1) = z$ (with $0 \leq z^2 \leq nR^2$ or equivalently $-\sqrt{n}R \leq z \leq \sqrt{n}R$) is given by

$$(\sqrt{nR^2 - z^2})^{n-1}.V_{n-1}$$

[68]To fit better with modern terminology, I use the word *ball*, though Gateaux and Lévy systematically use *sphere*.

[69]In fact, Gateaux started from a *continuous* function x. However, as Lévy explained to Fréchet in a long letter dated from 16 February 1919 (Letter 5 in [Barbut, 2004]), it is more natural to consider measurable functions, i.e. to work with the (now) usual space L^2. This is what he does in [Lévy, 1922].

[70]Gateaux considers this functional although it is naturally not continuous. The importance of continuity in Gateaux's considerations about infinite dimensional is not completely determined. It is likely that Gateaux would have improved the apparent incoherence in a subsequent rewriting of the paper. We shall see that Lévy fixed the question in [Lévy, 1922].

where V_n is the volume of the unit sphere in dimension n. V_n satisfies the induction formula $V_n = 2V_{n-1}\int_0^{\frac{\pi}{2}} \cos^n \theta d\theta$ and the mean of the functional U over the n-th section is

$$\frac{1}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n \theta d\theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(R\sqrt{n}\sin\theta)\cos^n \theta d\theta.$$

It is seen that the preponderant values for θ in the last integral are those around 0, and the first one is known to be asymptotically equivalent to $\sqrt{\frac{2\pi}{n}}$. Under *some regularity conditions* for f, the previous expression is therefore approximately equal to

$$\frac{1}{\sqrt{\frac{2\pi}{n}}} \int_{-\alpha}^{\alpha} f(R\sqrt{n}\sin\frac{\psi}{\sqrt{n}})\cos^n \frac{\psi}{\sqrt{n}}\frac{d\psi}{\sqrt{n}}$$

for any $\alpha > 0$ and sufficiently large n.

Using a Taylor expansion, and letting n, and then α go to infinity, the latter expression converges to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(R\psi)e^{-\frac{\psi^2}{2}} d\psi \tag{1}$$

defined by Gateaux as the integral (or the mean) of U over the sphere of the functional space. He asserted that this result can be generalized for functionals of the type

$$U(x) = \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_p f[x(\alpha_1), \ldots, x(\alpha_p), \alpha_1, \ldots, \alpha_p]$$

for which the mean value is given by

$$\frac{1}{(2\pi)^{\frac{p}{2}}} \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_p \int_{-\infty}^{+\infty} dx_1 \ldots \int_{-\infty}^{+\infty} dx_p f(Rx_1, \ldots, Rx_p, \alpha_1, \ldots, \alpha_p) \mathrm{e}^{-\frac{x_1^2+\cdots+x_p^2}{2}}. \tag{2}$$

The rigorous existence of the limit was not justified by Gateaux, as Lévy wrote to Fréchet in his letter from 12 February 1919. Obviously, for Gateaux, as Lévy himself wrote in the foreword of [Gateaux, 1919a], the present state of his papers was certainly not a final one. And in the long note Lévy added at the end of the article ([Gateaux, 1919a], p.67), he described the attempts made by Gateaux to obtain the limit in several situations. For Lévy, the priority was to fill the gap left by Gateaux and to try to obtain the existence of the mean value for the most general functionals.

The most general type of functionals U considered by Gateaux ([Gateaux, 1919a], p.52) is such that for any $\varepsilon > 0$, there is an n such that if x and y are two functions of the ball $\int_0^1 x(\alpha)^2 d\alpha \leq R^2$, assuming the same mean value over each subinterval $[\frac{i-1}{n}, \frac{i}{n}]$, $\mid U(x) - U(y) \mid < \varepsilon$. Following Gateaux, for such a functional, the mean value is given by its value at the center. It is therefore a case of an *harmonic* function.

Gateaux's condition was natural for him, as he had proved in [Gateaux, 1913b], that such a continuous functional U is uniformly approximated over the ball $\int_0^1 x(\alpha)^2 d\alpha \leq R^2$ by $U(y_n)$ where y_n belongs to the n-th section of the sphere and on each interval $[\frac{i-1}{n}, \frac{i}{n}]$ is equal to the mean value of x over this interval.

However, after he began to scrutinize Gateaux's paper, Lévy was convinced that Gateaux's condition was much too restrictive. As early as 16 February 1919[71], he mentioned the fact to Fréchet.

[71][Barbut, 2004], p.115.

And in the final version of his ideas on the question, in [Lévy, 1922] (p.277), he was even more drastic in his conclusions: under very general assumptions, a continuous functional takes *almost everywhere* the same constant value. Its mean is therefore obviously equal to this value. Lévy gives ([Lévy, 1922], p.275) a simple example illustrating this situation. Consider $U(x) = \varphi(r)$, where φ is a continuous function on $\mathbb{R}_+$ and $r^2 = \int_0^1 x^2(t)dt$. The volume of the ball $B_n(R)$ with radius R centered in 0 in $\mathbb{R}^n$ is proportional to R^n; hence, for any given $0 < \varepsilon < 1$, the quotient of the volumes of $B_n((1-\varepsilon)R))$ and $B_n(R)$ tends to 0, which means that when n grows, the volume is more and more concentrated close to the surface. Therefore, $\varphi(R)$ is essentially the only value assumed by φ in the ball counting for the estimation of the mean.

4.3. **Lévy's probabilistic interpretation.** I have already mentioned that in 1919, Lévy had his first contact with probability theory when he was asked to teach probability at the Ecole Polytechnique[72]. This was exactly the same period he was studying Gateaux's papers and preparing their publication. One may observe that probability theory never appears in the various notes presented by Lévy to the Paris Academy of Sciences as he progressed in his work on Gateaux ([Lévy, 1919a], [Lévy, 1919b], [Lévy, 1919c], [Lévy, 1921])[73]. Only when he wrote his book [Lévy, 1922] did he adopt probabilistic reasonings as relevant for his considerations about the mean in a functional space. As this happened the very year he published his first papers on probability, it seems that a kind of extraordinary junction occurred in Lévy's mind resulting in unifying his mathematical interests.

Let us try to understand how probability entered Lévy's considerations about the mean in functional spaces (Part Three of [Lévy, 1922]). Consider ([Lévy, 1922], p.266) in the space $\mathbb{R}^n$ the volume of the portion of the ball centered at 0 with radius $R\sqrt{n}$, comprised between the two hyperplanes $z = R\xi_1$ and $z = R\xi_2$, where z is the distance to a fixed hyperplane containing 0. The quotient of this volume and the total volume of the ball is equal to

$$\frac{\int_{\frac{\xi_1}{\sqrt{n}}}^{\frac{\xi_2}{\sqrt{n}}} \cos^n \theta d\theta}{\int_{-\pi/2}^{+\pi/2} \cos^n \theta d\theta}$$

which tends to

$$\frac{1}{\sqrt{2\pi}} \int_{\xi_1}^{\xi_2} \mathbf{e}^{-\frac{x^2}{2}} dx. \tag{3}$$

More generally, consider p hyperplanes containing 0 and call $z_1, z_2, \dots, z_p$ the distances to these hyperplanes. The volume of the intersection of p regions $R\xi_i' < z_i < R\xi_i''$ $(i = 1, 2, \dots, p)$ is a fraction of the total volume equal to

$$\frac{1}{(2\pi)^{p/2}} \int_{\xi_1'}^{\xi_1''} dx_1 \int_{\xi_2'}^{\xi_2''} dx_2 \dots \int_{\xi_p'}^{\xi_p''} dx_p \mathbf{e}^{-\frac{x_1^2+x_2^2+\cdots+x_p^2}{2}}.$$

This is, writes Lévy, a direct consequence of the independence of the random variables z_i, each following a Gaussian distribution according to the previous result. Lévy's proof is based on a geometrical approach which would become his typical trademark in numerous later works in

[72] For more details about this story, I refer the reader to [Barbut and Mazliak, 2008a].

[73] However, Lévy began to work on independent probabilistic questions at the same time. See in particular [Barbut, 2004], pp.40-44 about Lévy's investigations on stable distributions.

probability[74]. To prove the desired independence , writes Lévy, it is sufficient to prove that the conditions $z_i = R\xi_i, i = 1, 2, \ldots, p-1$ do not influence the distribution of z_p. The intersection of these conditions is a hyperspace H with dimension $n-p+1$, included in a hyperplane $r = kR$ (r being the distance between 0 and H). Now, the intersection of H and the ball of radius $R\sqrt{n}$ is a ball with dimension $n-p+1$ and radius $R\sqrt{n-k^2}$, equivalent to $R\sqrt{n-p+1}$ when n tends to infinity. Moreover, $n-p+1$ tends to infinity with n. Therefore, concludes Lévy, the distribution of z_p is given by the formula (3), hence the desired independence.

The probabilistic approach allowed Lévy to obtain Gateaux's formula (1) for the mean of the functional $U(x) = f[x(\tau)]$ in a direct way ([Lévy, 1922], p.278). x being in the ball with radius $R\sqrt{n}$, the probability of the event $R\xi_1 \leq x(\tau) \leq R\xi_2$ tends with n towards $\frac{1}{\sqrt{2\pi}}\int_{\xi_1}^{\xi_2} \mathrm{e}^{-\frac{\xi^2}{2}} d\xi$, so that the mean of U is given by (1). In the same way, the mean of $U(x) = \varphi(x(t_1), x(t_2), \ldots, x(t_p))$ is immediately obtained using the fact that $x(t_1), x(t_2), \ldots, x(t_p)$ are i.i.d. Gaussian variables ([Lévy, 1922], p.281). Probabilistic reasoning eventually enables to explain the concentration at the surface of the ball ([Lévy, 1922], p.283).

In Chapter VI ([Lévy, 1922], Part Three, p.421), Lévy studies the general question of the existence of the mean for a functional. As we have seen in the previous subsection, Lévy considered spatial continuity as too restrictive a condition. In this Chapter, he highlights that it is convenient not to look at the values of the function x, but at the probability distribution of these values.

As a basic example he considers the mean of the functional $U(x) = F(f)$ in the ball with radius R, where f is the probability distribution function (called by Lévy *fonction sommatoire*) of x over the space [0,1] equipped with Lebesgue measure λ[75]. Lévy's reasoning is as follows. If x belongs to the n-th section of the ball, it is a function constant in each interval $[\frac{i-1}{n}, \frac{i}{n}]$ with value x_i, such that $x_1^2 + x_2^2 \cdots + x_n^2 = nR^2$. For n sufficiently large, the x_i are independent Gaussian random variables with variance R. Moreover, the probability distribution function associated with this x is the frequency curve of the values $x_1, x_2, \ldots, x_2$ which is very close to the Gaussian distribution function with variance R (denoted by φ) by virtue of the law of large numbers. This allows Lévy to conclude ([Lévy, 1922], p.424) that the mean of U is equal to $F(\varphi)$.

As a generalization of the previous result, Lévy studies functionals U satisfying a condition which, though weaker than spatial continuity, guarantees a good approximation of the functional by its values on the n-th section. The most general property (called $\mathcal{H}$ by Lévy - [Lévy, 1922], p.424) he considers is the following : for each given $\varepsilon > 0$, there is a n such that, if x and y are two functions in the ball such that in every interval $[\frac{i-1}{n}, \frac{i}{n}]$ the probability distribution function of x and y is the same, $\mid U(y) - U(x) \mid < \varepsilon$. However, Lévy was not able to prove the result in all the desired generality but asserts that it is *reliable* ([Lévy, 1922], p.427).

As it is seen, probability reasoning is omnipresent in the Third Part of [Lévy, 1922]. Lévy was certainly conscious of the profound originality of his approach and desired to convince everyone of its interest. The complicated relations between the French prominent mathematicians (Borel and Hadamard in the first place) and probability theory has been considered in several studies

[74]This is the precise aspect that explains what Itô wrote later, about his difficult work to *translate* Lévy. *At that time*, writes Itô, *it was commonly believed that Levy's works were extremely difficult, since Lévy, a pioneer in the new mathematical field, explained probability theory based on his intuition. I attempted to describe Lévy's ideas, using precise logic that Kolmogorov might use* ([Itô, 1998]).

[75]This is to say that x is considered as a random variable on the probability space $[0, 1]$ with Lebesgue measure. Hence $f(\xi) = \lambda\{t \in [0, 1], x(t) \leq \xi\}$.

(see [Bru, 2003] and [Durand and Mazliak, 2011] and the references included for more details). It has been observed that from the very beginning of his interest in probability, Lévy felt himself unjustly despised for his choice[76], though he was comforted by Wiener's reaction to his approach (I shall come back on that point in the next subsection).
This disinterest of the leading French mathematicians in probability (Borel was the exception) may be an explanation why absolutely no reference to probability can be located in Gateaux's papers, even when he observed the remarkable appearance of the Gaussian distribution in the limit expression (1). Borel had proved the convergence of uniform measure on the ball of radius $R\sqrt{n}$ to the Gaussian measure when the dimension of the space tends to infinity in 1906[77]. In [Borel, 1906][78], Borel's interest was statistical mechanics, more precisely for Maxwell and Boltzmann's kinetic theory of gases. In his presentation, the spheres represent surfaces in the phase space of equal total kinetic energy. In a complement to his translation of Ehrenfests' paper on statistical mechanics in *Encyclopédie des Sciences Mathématiques* ([Borel, 1914b] p.273), Borel mentions studies about the n-dimensional sphere as the first example of mathematical research inspired by statistical mechanics. He even audaciously asserts that one should consider the results about surfaces and volumes in high dimensions as connected to statistical mechanics. However, in contrast to Maxwell who, in his fundamental paper in 1860, had emphasized the coincidence between the distribution law for the speeds of the particles and the distribution *governing the distribution of errors among observations by use of the so-called* least-squares method[79], Borel did not mention the point in [Borel, 1906]. When he mentioned it in [Borel, 1914a] (p.66), it was without any probabilistic interpretation. For Borel, the interesting fact in this appearance of the Gaussian distribution function was only that this distribution function is tabulated, which allows it to be used for computations.
It is probably the desire to explain to a large audience why probabilistic tools were useful that prompted Lévy to write a non-technical paper for the *Revue de Métaphysique et de Morale* ([Lévy, 1924]). Lévy exposes there the general ideas leading to his conception of the mean value, based on probability considerations over general sets[80]. As a basic example, he considers the situation of nonnegative integers (today called probabilistic number theory). If f is a function defined on $\mathbb{N}$ (f could typically be the indicator of a subset $A \subset \mathbb{N}$), the mean of f is defined as the limit of $\frac{1}{N}\sum_{k=1}^{N} f(k)$ when N tends to infinity. In particular, $P(A) = \lim_{N\to+\infty} \frac{1}{N}\mathrm{Card}\{n \in$

[76]On that topic, see in particular [Barbut and Mazliak, 2008b].

[77]The result, usually known today under the name *Poincaré's lemma*, has in fact nothing to do with Poincaré, according to Diaconis and Freedman [Diaconis, 1987]. Moreover, Stroock ([Stroock, 1994]) discovered that Mehler had already obtained the result in 1866 in a purely analytical context (see [Stroock, 1994], p.68, footnote 3 for an exact reference and comments).

[78]Reprinted as *Note I* in his book [Borel, 1914a].

[79][Maxwell, 1860], Prop.IV and following comments.

[80]Interestingly, Lévy asserts ([Lévy, 1924], p.149) that the article is the development of his last lecture of the Cours Peccot of 1919, meaning that the aforementioned 'junction' between probability and his studies in functional calculus appeared quite early in his mind. This is corroborated by his first letters to Fréchet ([Barbut, 2004], Letters 1-5 (before February 1919). If probability is never mentioned explicitly there, one may observe how gradually Lévy is closer to probabilistic reasoning. A good example is found in Letter 3 ([Barbut, 2004], p.73) where Lévy writes about his desire to find a way of expressing that functions u such that $\int u'^2$ are *less probable*.

$I\!N, n \in A\}$[81]. The paper includes a presentation of Gateaux's work on infinite dimensional integration, and the idea behind the extension to more general functionals. Lévy was probably rather satisfied with the picture he had provided in his paper, as he decided to reprint it as an appendix in his treatise of probability published the next year ([Lévy, 1925a]). Another attempt to disseminate his considerations on functional analysis was also done in 1924. Henri Villat asked Lévy to write a small booklet for his new series *Mémorial des Sciences Mathématiques*. [Lévy, 1925b] contains 56 pages and appears in fact as a survey of the book [Lévy, 1922]. Lévy updated his bibliography and Daniell's and Wiener's works were now quoted.

4.4. **Wiener measure : Daniell versus Gateaux's integrals.** As mentioned in the introduction, it is well beyond the scope of this article to provide a complete description of the fundamental works where Wiener built the first mathematical model of Brownian motion; on that topic, I refer the reader to Itô's comments in [Wiener, 1976], [Kahane, 1998] and to [Barbut, 2004], pp.54-60. The aim of this section is more modest : to try to explain how Wiener became acquainted with Gateaux's approach to integration and how he eventually used it in his epoch-making paper [Wiener, 1923].

In the second half of the 1910s, the British mathematician Percy J.Daniell (1889-1946), then holding a position at the Rice Institute in Houston, Texas was interested in extending Lebesgue integration to infinite dimensional spaces[82]. Daniell wrote two important papers [Daniell, 1917] and [Daniell, 1918] on the subject. His approach was to consider the integral as an operator on functions satisfying certain properties such as linearity and a monotone convergence theorem on a restricted class of functions T_0 and to prove that these properties allow one to extend integration to the class T_1 of limits of sequences in T_0. It can be seen that such a construction is directly inspired by Lebesgue[83] .

Wiener's first work on functionals [Wiener, 1920] appeared in 1920. Wiener proved there that Daniell's method can be applied to define the integral in the space of functionals, taking as basic T_0 a set of step functions for which the integral is defined as a mean. Probably shortly before publication, Wiener added the following footnote ([Wiener, 1920],p.67):

> The use of mean instead of integral is found in the posthumous papers of Gateaux (Bull. de la Soc.Math. de France, 1919). This was however unknown to me at the time I wrote this article.

We do not know exactly when Wiener was informed of the existence of Gateaux's works. A possible hypothesis is that he became aware of them during his journey in France in 1920 when he came to the Strasbourg International Congress and met Fréchet and Volterra.

The next year, Wiener published his first papers on Brownian motion. In the first one ([Wiener, 1921a]), he starts from Einstein's result : at time t the probability that the position of a particle on a line belongs to the interval $[x_0, x_1]$ has the form $\frac{1}{\sqrt{\pi ct}}\int_{x_0}^{x_1} \mathbf{e}^{-x^2/ct}dx$ where c is a constant (taken equal to 1 by Wiener, thanks to a good choice of units). Thus, if we consider as functional a function of the path $(x = f(t), 0 \leq t \leq 1)$, a natural question arises of defining the average value

[81]Therefore if one randomly draws a point from $I\!N$, there is, for instance, one chance over two that it is an even integer, a rather comforting result for the mind. . .

[82]A very complete description of Daniell's work and personality can be found in the paper [Aldrich, 2007].

[83]Lévy did not hide his moderate appreciation of Daniell's work on integration to Fréchet. He wrote to him *if nothing important has escaped me, Daniell has given not a definition of the integral but an extension of the notion of integral from a restricted domain to a larger one. That is a Lebesgue-kind work. (*[Barbut, 2004], *p.86)*

of the functional. Due to the independence of increments in the Brownian motion, asserts Wiener, it is reasonable to associate to a functional of the form $F = \Phi(f(t_1), \dots, f(t_n))$ depending only on the values of f at some finite number of values of t, a mean, denoted $A[F]$ by Wiener, defined by

$$A[F] = \frac{1}{\sqrt{\pi^n t_1(t_2 - t_1)\dots(t_n - t_{n-1})}} \dots$$
$$\dots \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \Phi(x_1, \dots, x_n) \mathbf{e}^{-\frac{x_1^2}{t_1} - \frac{(x_2-x_1)^2}{t_2-t_1} - \dots - \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}} dx_1 \dots dx_n.$$

In particular, observes Wiener, if $F(f) = f(t_1)^{m_1} \dots f(t_n)^{m_n}$, one may compute an explicit value for $A[F]$. Therefore, if a functional F is analytical in the sense of Volterra, which means that it can be decomposed as a sum of functionals of the type

$$\int_0^1 \dots \int_0^1 f(x_1) \dots f(x_n) \varphi_n(x_1, \dots, x_n) dx_1 \dots dx_n{}^{84},$$

the mean of F is defined as the sum of the series

$$\sum \int_0^1 \dots \int_0^1 A[f(x_1) \dots f(x_n)] \varphi_n(x_1, \dots, x_n) dx_1 \dots dx_n$$

when this series is convergent. Wiener's paper proves that, with this definition, the mean satisfies the classical properties of integrals such as linearity or possibility of exchanging infinite summation and integration. Wiener quotes Gateaux ([Wiener, 1921a], Note 1, p.260) for having proposed using analytical functionals in the definition of the mean of a functional. As we have seen, it is true that Gateaux had such an idea in mind from the very beginning (see his programmatic letter to Borel in Appendix 1), but contrary to Wiener's assertion, the idea does not seem to be explicit in [Gateaux, 1919a]. Wiener adds that Gateaux's definition is, however, not well adapted to the treatment of Brownian motion.

Wiener published his second study ([Wiener, 1921b]) in the next issue of the Proceedings of the National Academy of Sciences. The aim of this new paper was to show that the use of the definition of the mean provided in [Wiener, 1921b] allowed one to obtain a direct proof (moreover under somehow lighter hypotheses) of Einstein's formula for the mean quadratic displacement of the Brownian particle in a viscous medium. Once again, Gateaux is mentioned as having proposed another construction of the mean

> To determine the average value of a functional, then seems a reasonable problem, provided that we have some convention as to what constitutes a normal distribution of the functions that form its arguments. Two essentially different discussions have been given on this matter: one, by Gateaux, being a direct generalization of the ordinary mean in n-space; the other, by the author of this paper, involving considerations from the theory of probabilities. ([Wiener, 1921b], p.295)

During the Summer of 1922, Wiener came again to France and met Lévy for the first time during his vacation in Pougues les Eaux, a spa in central France, and discussed Lévy's book on functional analysis. Lévy narrates the meeting in his autobiography, where he emphasizes that Wiener was almost the only one who immediately recognized the depth of Part III of his book [Lévy, 1922]

[84] Wiener considers in fact a generalization of this situation where the functionals are defined by means of Stieltjès integrals.

([Lévy, 1970], p.86) - and also on page 65). He adds he had reasons to think that this third part was the origin of his memoir [Wiener, 1923] on Brownian motion.
Indeed, in the introduction of [Wiener, 1923], Wiener pays full tribute to Lévy

> The present paper owes its inception to a conversation which the author had with Professor Lévy in regard to the relation which the two systems of integration in infinitely many dimensions - that of Lévy and that of the author - bear to one another. For this indebtedness the author wishes to give full credit. ([Wiener, 1923], p.132)

Gateaux is now clearly treated by Wiener only as a precursor, and Lévy has become the major source of inspiration. Besides, Wiener wrote ([Wiener, 1923], p.132) that Gateaux had begun investigations on integration in infinitely many dimensions which had been *carried out by Lévy* in [Lévy, 1922].
In [Wiener, 1923], Wiener reconsidered the results of his previous papers on Brownian motion. Contrary to what he had done in [Wiener, 1921a] where the mean of a functional $F = \Phi(f(t_1), \ldots, f(t_n))$ of the trajectory was given *a priori*, he now used Lévy's studies of the n-dimensional sphere and the Gateaux-Lévy definition of the mean as limit of the means over the n-th sections in order to :
1) *deduce* the Gaussian expression of the semi-group[85]
2) *define* the related measure on the space of continuous functions (Wiener measure)
3) *prove* the value of the mean of the aforementioned functional he had postulated in his previous works[86]
4) *derive* the expression of the mean of an analytic functional with a new proof[87].
Section 10 of [Wiener, 1923] is devoted to prove that, for the functionals previously considered, Daniell's extension of the mean Wiener had introduced in [Wiener, 1920] gives the same value to the integral[88].
Finally observe that Wiener's paper is not absolutely conclusive about the use of Daniell's versus Gateaux-Lévy's approach, though I can certainly interpret Wiener's choice to write the paper starting from the latter as recognition of its more intuitive character. Besides, it is well known that Lévy was never a great supporter of abstract constructions of Brownian motion. In his autobiography ([Lévy, 1970], p.98), Lévy, who was not shy on emphasizing his missed opportunities, regretted how he let Wiener get ahead of him in the construction of Brownian motion though all the necessary material was in [Lévy, 1922]. Lévy did sometimes slightly exaggerate his own role (as for example when he wrote about Kolmogoroff's *Grundbegriffe* ([Lévy, 1970], p.68)). In the case of Brownian motion however, one can understand his regrets.
The geometric approach to Brownian motion was quite fertile in 20th Century. McKean ([McKean, 1973]) has exposed how thinking of the Wiener measure as a uniform distribution over the *infinite dimensional sphere of radius* $\sqrt{\infty}$, a direct consequence of Lévy's considerations in [Lévy, 1922], has been successfully used by Japanese mathematicians in the 1960s to describe the geometry of

[85]([Wiener, 1923], pp. 136-137 - I use modern terminology, of course).
[86][Wiener, 1923], p.153
[87][Wiener, 1923], p.165
[88]The construction of the Wiener measure via Daniell's extension is tightly related to the theorem of extension Kolmogorov would provide 10 years later in his *Grundbegriffe* ([Kolmogorov, 1933]). On that topic, consult [Shafer and Vovk, 2006], in particular section 5.1, p.87.

Brownian motion. In another direction, in 1969, Gallardo ([Gallardo, 1969]) made the observation that *Poincaré's lemma* could be connected with the fact that if $X^n(t) = (X_1(t), \dots, X_n(t))$ is an n-dimensional Brownian motion issuing from 0, if one denotes by T_n the first passage time of X^n on the sphere centered at 0 and with radius $\sqrt{n}$, then $T_n \to 1$ in probability and $X^n(T_n)$ follows the uniform distribution on the n-dimensional sphere of radius $\sqrt{n}$. Yor later developed these considerations (see [Yor, 1997]).

CONCLUSION

It has often been said that after World War 1, the French Grandes Ecoles, the Ecole Normale especially, were crowded with the ghosts of the students from the 1910s who disappeared during the conflict. Of course these dead of the Great War were essentially very young men who had scarcely finished their graduate studies and whose names are hardly known to us today. René Gateaux, who died at the age of 25 in October 1914, is an example both representative and exceptional of the student victims of the war–exceptional because, despite being very young, he left scientific work that could be carried on by others.

Bourbaki, when he eventually added some words about probability theory in the chapter devoted to integration in non-locally compact spaces of a late edition of his *Eléments d'histoire des mathématiques* ([Bourbaki, 1984], pp.299-302)[89], mentioned the path linking Borel's consideration on kinetic theory of gases to the Wiener measure with Gateaux's and Lévy's works as fundamental steps.

Though uncompleted, Gateaux's mathematical studies were recovered and extended by Paul Lévy for whom they became a catalyst for a renewal of his scientific interests in probability. It is due to Lévy's work of editing and extension that today we remember Gateaux.

Acknowledgement: I want to express my deep thanks to the people who kindly contributed to the preparation of this paper. Firstly, to M.Pierre Gateaux, a distant relative of René Gateaux, for his warm welcome in Vitry and his help in my research for René's familial background, and to Gilbert Maheux, a local historian in Vitry le François for his enthusiastic help. In the course of this research, I have visited several archives. I would like to thank Giorgio Letta of the mathematical department in Pisa and member of the Accademia dei Lincei for his efficient help, and also those responsible for the archives in Rome for their help and kindness. Florence Greffe and the staff of the archives of the Academie des Sciences in Paris helped me in obtaining permission to consult Gateaux's manuscripts deposited at the Académie. Valuable assistance was provided by Francoise Dauphragne at the archives of the Ecole Normale Supérieure from whom I obtained a photograph of the 1907 pupils. And, last but not least, through a visit to the archives of the Paris Archdiocese and the help of Father Ph. Ploix, I discovered the only picture I know of where Gateaux is explicitly identified. I thank also my colleagues Olivier Guédon, Bernard Locker, Stéphane Menozzi and Marc Yor for several interesting discussions relating to Gateaux's life and work. I owe to John Aldrich and Daniel Denis the transformation of the original version of the text into what can be called real English. Finally, I am glad to mention the two anonymous referees who were confronted with a first version of this work and provided numerous suggestions which, I hope, have largely improved the paper.

[89] The complicated story of Bourbaki's attitude to integration, and in particular of Dieudonné's resistance to abstract integration is well known and presented in detail in [Schwartz, 1997]. I shall not make further comments on it here.

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APPENDIX 1 : GATEAUX'S PROGRAMMATIC LETTER TO BOREL -12 APRIL 1913

Bar-le-Duc, le 12 Avril 1913

Monsieur,

J'ai fait il y a quelques jours la demande d'une bourse David Weill. La circulaire adressée au proviseur indique la forme de cette demande, et porte qu'elle doit être faite avant le 1er Mai.

Voici les questions que je me propose d'étudier:

Fonctionnelles analytiques :

a- M.Volterra a tout d'abord appelé fonctionnelle analytique la somme de la série

$$U(f) = \sum_n^{0 \ldots \infty} \int_a^b K_n(x_1, \ldots, x_n) f(x_1) \ldots f(x_n) dx_1 \ldots dx_n.$$

M.Fréchet (Annales de l'Ecole Normale Sup[e], mai 1910) en a donné ne définition plus générale:

$$(1)U(f) = U_0(f) + U_1(f) + \cdots + U_n(f) + \dots$$

$U_0, U_1, \dots, U_n, \dots$ étant des fonctionnelles de f homogènes et d'ordre 0,1, ..., n . Cette définition est analogue à la définition des fonctions analytiques de plusieurs variables, d'après Weierstrass.

Je me propose d'abord d'étudier de plus près le développement (1), en particulier l'expression analytique des U_n.

b- On peut donner une définition des fonctionnelles analytiques analogue à la définition des fonctions analytiques de Cauchy.

Soit $f(\alpha) = f_1(\alpha) + if_2(\alpha)$ la fonction variable indépendante, définie pour α réel $a \leq \alpha \leq b$.

Soit $U(f) = U_1(f_1, f_2) + iU_2(f_1, f_2)$ une fonctionnelle continue par rapport à f c'est-à-dire telle qu'on puisse trouver η tel que $\mid f - \varphi \mid < \eta$ entraîne $\mid U(f) - U(\varphi) \mid < \varepsilon$.

Donnons à f la variation $\delta f = \delta f_1 + i\delta f_2$. U prend la variation δU.

Nous dirons que U est fonctionnelle analytique de f si δU est une fonctionnelle linéaire de δf.

Ce qu'on écrit : (2) $\delta U = \int_a^b \delta f(\alpha) da(\alpha)$

$a(\alpha) = a_1(\alpha) + ia_2(\alpha)$, fonction à variation bornée indépendante de $\delta f(\alpha)$. (Riesz, CR - 29 Nov. 1909).

Dans le cas où U admet, en outre, pour chaque valeur d'α , les dérivées (au sens de M.Volterra) par rapport à f_1, f_2, la condition (2) s'écrit

$$U'_{1,f_1}(f_1, f_2; \alpha) = U'_{2,f_2} \qquad U'_{1,f_2} = -U'_{2,f_1}$$

c- Les fonctionnelles analytiques, définition (1), vérifient la condition (2).

Je me propose, réciproquement, d'étudier les fonctionnelles analytiques, définition (2), et d'examiner si on peut les développer en séries (1).

Pour cela, la voie naturelle paraît être l'extension de l'intégrale de Cauchy.

Intégration d'une fonctionnelle.

On est alors conduit à définir l'intégrale d'une fonctionnelle continue réelle dans un champ fonctionnel réel.

Parmi les définitions qu'on peut en donner, voici la plus simple:

Soit la fonction variable indépendante réelle $f(\alpha); 0 \leq \alpha \leq 1$, U une fonctionnelle réelle continue de f définie dans l'ensemble des fonctions $0 \leq f \leq 1$, continues ou admettant un nombre fini de discontinuités.

Bornons nous à définir l'intégrale de U dans le champ des fonctions $0 \leq f \leq 1$. Divisons l'intervalle $(0, 1)$ en n intervalles, les limites de ces intervalles partiels étant considérées comme appartenant à l'un des deux qui lui sont adjacents.

Considérons la fonction f égale dans chaque intervalle partiel aux nombres $f_1, f_2, \dots f_n$ compris entre 0 et 1. $U(f)$ est une fonction des n variables $f_1 \dots f_n$: $U_n(f_1, \dots, f_n)$.

Considérons l'expression :

$$(3) I_n = \int_0^1 \int_0^1 \dots \int_0^1 U_n(f_1, \dots, f_n) df_1 \dots df_n$$

Faisons augmenter n indéfiniment, chaque intervalle tendant vers 0. Supposons que I_n tende vers une limite I indépendante des modes de division choisis. Nous dirons que I est l'intégrale de U dans le champ $0 \leq f \leq 1$.

On peut également en donner une définition ne faisant intervenir que les fonctions f continues.

La limite I existe en particulier pour les fonctionnelles analytiques, définition (1), holomorphes dans le champ $\mid f \mid < 1 + \varepsilon$.

Je me propose de rechercher si la limite I existe, U étant une fonctionnelle continue, ou s'il est nécessaire d'ajouter une autre hypothèse à celle de la continuité.

La définition de l'intégrale peut sans doute être étendue aux fonctionnelles présentant certaines discontinuités.

Enfin on passe de l'intégration dans un champ réel à l'intégration dans un champ imaginaire, et on peut en particulier étudier l'intégrale de Cauchy.

Applications:

Les applications de l'extension précédente sont certainement assez nombreuses. J'indique par exemple : l'extension aux fonctionnelles de certaines propriétés de résidus des intégrales multiples, de la formule de Lagrange démontrée par Poincaré dans le cas des fonctions de 2 variables (Acta mathematica, tome 3).

Tels sont les problèmes que j'envisage actuellement et dont je désire continuer l'étude l'an prochain à Rome.

Comptant sur votre appui, je vous prie d'agréer, Monsieur, avec mes remerciements, l'expression de mes sentiments respectueux.

R.Gateaux

Professeur au lycée de Bar-le-Duc

Translation

Bar-le-Duc, April 12th, 1913

Sir,

I applied for a David Weill grant some days ago. The circular letter addressed to the principal indicates the nature of this application, and mentions that it must be presented before May 1st.

Here are the questions I propose to study :

Analytical functionals

a) M Volterra has in the first place called analytical functional the sum of the series

$$U(f) = \sum_n^{0 \dots \infty} \int_a^b K_n(x_1, \dots, x_n) f(x_1) \dots f(x_n) dx_1 \dots dx_n.$$

M.Fréchet (Annales de l'Ecole Normale Sup^{e}, May 1910) has given a more general definition :

$$(1) \qquad U(f) = U_0(f) + U_1(f) + \dots + U_n(f) + \dots$$

$U_0, U_1, \ldots, U_n, \ldots$ being functionals of f homogeneous and with order $0, 1, \ldots, n$. This definition is analogous to the definition of analytical functionals of several variables, following Weierstrass.

I first propose to study more closely the development (1), in particular the analytical expression of the U_n

b- One may give a definition of analytical functionals analogous to the definition of Cauchy's analytical functions

Let $f(\alpha) = f_1(\alpha) + if_2(\alpha)$ be the independent variable function, defined for a real $\alpha, a \leq \alpha \leq b$.

Let $U(f) = U_1(f_1, f_2) + iU_2(f_1, f_2)$ be a functional continuous with respect to f i.e. such that one may find an η such that $\mid f - \varphi \mid < \eta$ implies $\mid U(f) - U(\varphi) \mid < \varepsilon$.

Let us give the variation $\delta f = \delta f_1 + i\delta f_2$ to f. U has variation δU.

We shall say that U is an analytical functional of f if δU is a linear functional of δf.

Which may be written : (2) $\quad \delta U = \int_a^b \delta f(\alpha) da(\alpha)$

$a(\alpha) = a_1(\alpha) + ia_2(\alpha)$, function with bounded variation independent of $\delta f(\alpha)$. (Riesz, CR - 29 Nov. 1909).

In the case where U admits, moreover, for each value of α , derivatives (in M.Volterra's sense) with respect to f_1, f_2, condition (2) can be written as

$$U'_{1,f_1}(f_1, f_2; \alpha) = U'_{2,f_2} \qquad U'_{1,f_2} = -U'_{2,f_1}$$

c- Analytical functionals, definition (1), satisfy the condition (2).

I propose, reciprocally, to study the analytical functionals, definition (2), and to examine if they may be developed in series (1).

To achieve this, the most natural way seems the extension of the Cauchy integral.

Integration of a functional.

One is thus lead to define the integral of a real continuous functional in a real functional field.

Among the definitions one may provide, here is the simplest:

Consider the real independent variable function $f(\alpha); 0 \leq \alpha \leq 1$, U a real continuous functional of f defined in the set of functions $0 \leq f \leq 1$, continuous or having a finite number of discontinuities.

Let us restrict ourselves to the definition of the integral of U in the field of the functions $0 \leq f \leq 1$. Let us divide the interval (0,1) into n intervals, the limits of these partial intervals being considered as belonging to one of the two adjacent intervals

Consider now the function f equal in each partial interval to the numbers $f_1, \ldots, f_n$ which are between 0 and 1. $U(f)$ is a function of the n variables $f_1 \ldots f_n : U_n(f_1, \ldots, f_n)$.

Let us consider the expression

$$(3) I_n = \int_0^1 \int_0^1 \ldots \int_0^1 U_n(f_1, \ldots, f_n) df_1 \ldots df_n$$

Suppose that n increases to infinity, each interval converging to 0, and suppose that I_n tends to a limit I independent of the chosen divisions. We shall say that I is the integral of U over the field $0 \leq f \leq 1$.

One may also give a definition using continuous functions f only.

The limit I exists in particular for analytical functionals, definition (1), holomorphic in the field $\mid f \mid < 1 + \varepsilon$.

I propose to check whether limit I exists, U being a continuous functional, or if it is necessary to add an extra hypothesis to that of continuity.

The definition of the integral may probably be extended to functionals having some discontinuities.

At last one goes from integration in a real field to integration in an imaginary field, and one may in particular study Cauchy's integral.

Applications:

The applications of the previous extension are certainly rather numerous. I mention, for instance : the extension to functionals of certain properties of residues of multiple integrals, of Lagrange's formula proved by Poincaré in the case of functions of 2 variables (Acta mathematica, tom 3).

Here are the problems I presently foresee and of which I desire to study further next year in Rome.

Counting on your support and thanking you, I am, Sir, yours respectfully

R.Gateaux

Professor at Bar-le-Duc lycée

APPENDIX 2 : GATEAUX'S LAST LETTER FROM THE FRONT TO VOLTERRA - 25 AUGUST 1914

Buissoncourt (Meurthe-et Moselle), le 29 août 1914

Monsieur le Sénateur,

Je vous remercie beaucoup de votre lettre que je viens seulement de recevoir, dans les plaines de Lorraine où nous passons les jours et les nuits, au son du canon.

Je crois que la traduction des Leçons sur les fonctions de lignes se fera, comme vous le dites, longtemps attendre. Et quant à votre Mémoire sur les fonctions permutables, que j'aurais lu avec tant d'intérêt, Dieu sait quand je pourrai l'étudier!

Je tiens à vous dire combien j'ai été heureux d'apprendre que l'Italie, non seulement reste neutre, mais encore se rapproche de la France. Tous les français y ont été très sensibles, et ont vivement apprécié l'attitude de l'Italie. Puisse ce geste pousser nos deux pays à se connaître mieux et à se rapprocher davantage!

Le service postal fonctionne très irrégulièrement. Je compte vous écrire bientôt de nouveau et j'espère qu'une au moins de mes lettres vous parviendra. J'espère qu'elle vous trouvera en bonne santé ainsi que votre famille. Pour mon compte, je supporte parfaitement les fatigues de la campagne. Veuillez me rappeler au bon souvenir des professeurs que j'ai eu l'honneur de connaître à Rome. Veuillez présenter à Madame Volterra mes plus respectueux hommages et agréer l'expression de mes sentiments respectueux.

R.Gateaux, Lieutenant au 269ème RI, 139ème brigade, 70ème division de réserve, par Troyes (Aube)

Translation

Buissoncourt (Meurthe-et Moselle), August 29th, 1914

Mr Senator,

I deeply thank you for your letter I have just received, in the fields of Lorraine where we live days and nights under the sound of canons.

I trust that the translation of the *lectures on functions of lines* will be, as you mention, delayed for a long time. And as for your *Memoir on permutable functions*, that I would have read with so much interest, God knows when I will have the chance to study it!

I wanted to tell you how happy I was to learn that Italy, not only remains neutral, but is also more and more favorable to France. All the French people have been sensitive on that point, and have greatly appreciated Italy's attitude. May that act encourage our two countries to know each other better and to become closer!

The mail service works with much irregularity. I intend to write to you soon again and I hope that at least one of my letters will reach you. I hope it will find you and your family in good health. As for me, I bear perfectly the fatigue of the battle.

Please recall me to the professors I had the honor to meet in Rome. Will you transmit my most respectfully regards to Mrs Volterra and accept my most respectful sentiments.

R.Gateaux, Lieutenant at the 269th RI, 139th brigade, 70th reserve division, by Troyes (Aube)

Appendix 3 : Postcard from the Proviseur of the Lycée in Bar le Duc

Lycée de Bar le Duc, le 7 décembre 1914

Monsieur,

Notre regretté collègue, M.Gâteaux [!], n'a plus que sa mère. Elle habite actuellement à Vitry-le-François, rue de la Petite Sainte. Dans le deuil si cruel qui la frappe, elle serait très sensible à vos condoléances et ce serait une joie douloureuse pour elle de lire votre appréciation de la grande valeur intellectuelle de son cher disparu.

J'ai vu M.Gateaux, pour la dernière fois, le 20 juillet. Nous ne pensions pas à la guerre, ni l'un ni l'autre. Il me parla longuement de son séjour à Rome, et me dit que sa thèse de doctorat était presque terminée. Il a donc laissé des travaux qui mériteraient d'être publiés et il me semble que vous pourriez en entretenir sa mère.

La mort de notre jeune collègue a douloureusement ému tous ses collègues du Lycée de Bar qui avaient su apprécier sa belle intelligence, la franchise de son caractère et le charme de sa modestie. Il a fait vaillamment tout son devoir, jusqu'au bout, mais c'est grand'pitié qu'il ne lui ait pas été donné de vivre toute sa vie.

Veuillez agréer, Monsieur, l'assurance de mes sentiments les plus distingués.

Le Proviseur, M.Chemin

Translation

Lycée de Bar le Duc, Decembre 7th, 1914

Sir,

Our late lamented colleague M.Gateaux has only his mother left. She lives now in Vitry-le-François, rue de la Petite Sainte. In the so cruel mourning which strikes her, she would be very touched by your condolences and it would be a painful joy for her to read your appreciation of the great intellectual value of her dear loss.

I saw M.Gateaux for the last time on July 20th. Neither of us thought about war. He told me at length about his stay in Rome, and told me that his doctoral thesis was almost completed. He has therefore left works which merit being published and I think you could speak to his mother about it.

The death of our young colleague has painfully moved all his colleagues of the Lycée in Bar who had been able to appreciate his fine intelligence, the frankness of his character and the charm of his modesty. He valiantly did his duty up to the end, but it is a great pity that he was not given the opportunity to live a full life.

Respectfully yours,

The principal, M.Chemin