On an open problem regarding totally Fenchel unstable functions

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Let E be a nontrivial real Banach space and $f, g : E \to \overline{\mathbb{R}}$ two arbitrary convex functions. We say that f and g satisfy *stable Fenchel duality* if for all $x^* \in E^*$, there exists $z^* \in E^*$ such that

$$(f+g)^*(x^*) = f^*(x^* - z^*) + g^*(z^*).$$

If this property holds for $x^* = 0$, then f and g satisfy the classical *Fenchel duality*. The pair f, g is *totally Fenchel unstable* (see [2]) if f and g satisfy Fenchel duality but

$$y^*, z^* \in E^* \text{ and } (f+g)^*(y^* + z^*) = f^*(y^*) + g^*(z^*) \Longrightarrow y^* + z^* = 0.$$

Obviously, stable Fenchel duality implies Fenchel duality, but the converse is not true (see the example in [1], pp. 2798-2799 and Example 11.1 in [2]). Nevertheless, each of these examples (which are given in $\mathbb{R}^2$) fails when one tries to verify total Fenchel unstability. Surprisingly, in the finite dimensional case, it is still an open question if there exists a pair of functions which is totally Fenchel unstable. In the infinite dimensional setting this problem receives an answer, due to the existence of extreme points which are not support points of certain convex sets. In [2] the author has proposed an example (Example 11.3) of a pair f, g which is totally Fenchel unstable.

In this talk we give an answer to the Problem 11.5 posed by Stephen Simons in his book [2] in connection to this example.

References

[1] Boţ, R.I., Wanka G. (2006): *A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces*, Nonlinear Analysis: Theory, Methods & Applications 64 (12), 2787–2804.

[2] Simons, S.: *From Hahn-Banach to Monotonicity*, Springer-Verlag, Berlin, in press.