Closedness Type and Subdifferential Regularity Conditions for Conjugate Duality in Infinite Dimensional Spaces

Gert Wanka
Chemnitz University of Technology, Faculty of Mathematics, D-09107 Chemnitz, Germany

Let X and Y be two separated locally convex spaces, Y partially ordered by a nonempty closed convex cone $C \subseteq Y$ having C^* as dual cone, $U \subseteq X$ a nonempty closed convex subset, $g : X \to Y^\bullet = Y \cup \{\infty_Y\}$ a proper C-convex and C-*epi*-closed function such that $\mathcal{A} = U \cap g^{-1}(-C) = \{x \in U : g(x) \in -C\} \neq \emptyset$. Further, let be $f : X \to \overline{\mathbb{R}}$ a proper convex and lower semicontinuous function satisfying $\mathcal{A} \cap dom(f) \neq \emptyset$.

We consider the convex optimization problem

$$(P) \qquad \inf_{x \in \mathcal{A}} f(x)$$

and additionally the linearly perturbed problem

$$(P_p) \qquad \inf_{x \in \mathcal{A}} [f(x) + \langle p, x \rangle]$$

with the linear continuous functional $p \in X^*$.

We associate to (P_p) the well known Lagrange as well as different so-called Fenchel-Lagrange dual problems as e.g.

$$(D_p) \qquad \sup_{\substack{\lambda \in C^* \\ \alpha, \beta \in X^*}} \{-f^*(\beta) - \sigma_U(-p - \alpha) - (\lambda g)^*(\alpha - \beta)\},$$

$f^* : X^* \to \overline{\mathbb{R}}$, $f^*(\beta) = \sup_{x \in X} \{\langle \beta, x \rangle - f(x)\}$ denotes the conjugate function to f, $(\lambda g)(x) = \langle \lambda, g(x) \rangle$ for $x \in dom(g)$ and $(\lambda g)(x) = +\infty$ otherwise, $\sigma_U : X^* \to \overline{\mathbb{R}}$ is the support function with respect to U.

For $p = 0$ (D_0) turns out to be the Fenchel-Lagrange dual problem (D) to (P). We characterize strong, stable strong and total duality by means of the weakest regularity conditions known so far. We recall that by strong duality we understand the situation when the optimal objective values of the primal and dual problem coincide and the dual problem has a solution. When there is strong duality and the primal problem has an optimal solution, too, we speak of total duality. With stable strong duality we refer to the case when there is strong duality for all the optimization problems (P_p), $p \in X^*$, and their corresponding dual problems.

We consider different regularity conditions of closedness type, e.g.

$$(C_1(f,\mathcal{A})) \qquad epi(f^*) + epi(\sigma_U) + \bigcup_{\lambda \in C^*} epi((\lambda g)^*) \quad \text{is closed.}$$

As a typical result we point out that $(C_1(f,\mathcal{A}))$ is fulfilled if and only if there is stable strong duality for (P) and (D), i.e. one has strong duality for (P_p) and (D_p) for all $p \in X^*$.
Further, within the paper we offer some new subdifferential-type regularity conditions ensuring total duality for Lagrange as well as Fenchel-Lagrange duality. In particular instances our regularity conditions turn into some constraint qualifications recently published by other authors (cf. e.g. [2], [3]).

References

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