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Rational values of the Riemann zeta function

It is classical that the values

$$\zeta(0)=-\frac{1}{2},\ \zeta(-1)=-\frac{1}{12},\ \zeta(-2)=0,\ \zeta(-3)=\frac{1}{120},\ \zeta(-4)=0,\ldots,\ \zeta(-11)=\frac{691}{32760},\ldots$$

at non-positive integers are all rational. By contrast the values

$$\zeta(2)=\frac{\pi^2}{6},\ \zeta(4)=\frac{\pi^4}{90},\ldots,\ \zeta(12)=\frac{691\pi^{12}}{638512875},\ldots$$

are all irrational thanks to the transcendence of π. Apéry proved in 1978 that $\zeta(3)$ is irrational, but we still do not know that $\zeta(5)$ is irrational. However Ball and Rivoal proved in 2001 that the number of irrationals among $\zeta(3),\zeta(5),\zeta(7),\ldots,\zeta(2n+1)$ is at least $c\log n$ for some $c>0$ independent of n.

Even less is known about $\zeta(x)$ at rational x, say with $2<x<3$. We sketch a proof that the number of these x in $\frac{1}{n}\mathbf{Z}$ such that $\zeta(x)$ is also in $\frac{1}{n}\mathbf{Z}$ is at most $C(\log n)^2$ for some C also independent of n.