Curriculum Vitae

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Contents

Raffaele Dario Marcovecchio, born on 8th December 1974 at Foggia (Italy), italian citizen.

1 Current position

July 1, 2012 - June 30, 2013: INdAM research grant "Ing. Giorgio Schirillo" at the

✉ Dipartimento di Matematica Largo B.Pontecorvo, 5 56127 Pisa Italy

2 Qualifications

Master's degree in Mathematics at the University of Pisa, under the direction of Prof. C. Viola, obtained on 25.11.1999.

PhD in Mathematics at the University of Pisa. Thesis under the supervision of Prof. C.Viola, defended on 15.11.2004 before the Committee as follows: Prof. F.Amoroso, Prof. R.Dvornicich, Prof. A.Perelli, Prof. H.P.Schlickewei, Prof. C.Viola.

3 Former scientific and educational activities

October 1, 2006 - August 31, 2007 PostDoc, with teaching duties at the " Laboratoire de Mathématiques Nicolas Oresme"at the University of Caen, France. Official name of the position:

ATER: Attaché Temporaire d'Enseignement et de Recherche.

September 1, 2007 - 31 August 2008, ATER at the "Institut Fourier" of the University of Grenoble I, France.

September 1, 2008 - 31 August 2011 PostDoc at the Faculty of Mathematics at the University of Vienna, Austria in the Combinatorics Group headed by Prof. C. Krattenthaler.

4 Computer skills

Windows XP, Unix, Mac OS X, Latex, Maple, Mathematica, Pari/GP.

5 Languages

Italian: mother tongue. French: discreet. English: good. German: basic.

6 Publications

[1] PhD thesis "Alcuni Problemi di Approssimazione Diofantea", Pisa, Italy.

[2] Raffaele Marcovecchio, *Determinanti polinomiali-esponenziali*, Boll. Unione Mat. Ital. (8) **7**-B (2004), 713–730.

[3] Raffaele Marcovecchio, *Linear independence of linear forms in polylogarithms*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **5** (2006), 1–11.

[4] Raffaele Marcovecchio, *The Rhin-Viola method for* $\log 2$, Acta Arith. **139** (2009), 147–184.

[5] Raffaele Marcovecchio and Carlo Viola, *Irrationality and non-quadraticity measures for logarithms of algebraic numbers*, to appear in Journal of Australian Mathematical Society.

[6] Raffaele Marcovecchio, *Symmetry in Legendre-type polynomials and Diophantine approximation of logarithms*, MFO no.22/2012.

7 Teaching experiences

Academic year 2006/2007, University of Caen:

First semester:

Algebra, 36 hours of exercises for the fourth year of Master's degree in Fundamental Mathematics and Mathematics for Computer science.

Further Mathematics, 8 hour of lectures and 20 hours of exercises for the first year of Bachelor's degree in Applied Mathematics and Social Sciences.

Second semester:

Statistics, 28 hours of exercises for the first year of Bachelor's degree in Biology.

A.y. 2007/2008, University of Grenoble:

First semester:

Linear Algebra and Analysis, 52 hours of exercises for the first year of Bachelor's degree in Geology.

Second semester:

Introduction to dynamical systems and modeling, 39 hours of exercises for the first year of Bachelor's degree in Biology.

8 Seminars delivered at international conferences

8/9/2006: Linear independence of linear forms in polylogarithms, CIRM, Luminy (France).

20/6/2008: La méthode de Rhin-Viola pour $\log 2$, conference in honor of Georges Rhin: Méthodes algorithmiques en théorie des nombres, Paul Verlaine University, Metz, France.

10/3/2010: Rational approximations to the logarithm and its square, Italy-India Conference on Diophantine and Analytic Number Theory, CRM Ennio DeGiorgi, Pisa, Italy.

24/4/2012: Symmetry in Legendre-type polynomials and Diophantine approximation of logarithms, MFO Conference *Diophantische Approximationen*, Oberwolfach, Germany, organized by Y.Bugeaud and Yu.Nesterenko.

31/07/2012: The irrationality measure of $\zeta(2)$ revisited, International Conference "Diophantine Analysis", Russia, organized by R.Akhunzhanov, N.Moshchevitin and Yu.Nesterenko.

1/10/2012: On the irrationality measures of values of special functions, ERC Research Period on Diophantine Geometry, CMR Ennio De Giorgi, SNS, Pisa, Italy. Scientific Committee: E.Bombieri, D.Masser, L.Szpiro, G.Wüstholz, S.-W. Zhang. Organizing Committee: P.Corvaja, R.Dvornicich, U.Zannier.

9 Recent seminars

11/6/2008: Approximations simultanées de $\log 2$ et $\log^2 2$, Grenoble, France.

24/6/2008: Intégrales doubles complexes et $\log 2$, Lyon, France.

8/7/2008: Il metodo di Rhin-Viola per $\log 2$, Pisa, Italy.

23/1/2009: Nouvelles mesures d'irrationalité et de non-quadraticité de $\log 2$, Caen, France.

2/2/2012: Approssimazioni mediante razionali ed irrazionali quadratici di logaritmi di numeri razionali, Parma, Italy.

10 Participation in conferences

Selecta:

28 June to 6 July 2000, Diophantine Approximation, CIME Foundation, Cetraro, Italy. Organized by F.Amoroso and U.Zannier.

11 to 18 July 2002, Analytic Number Theory, Cetraro, CIME Foundation. Organized by A.Perelli and C.Viola.

13 to 15 November 2003, Second Italian Conference on Number Theory, Parma, Italy. Organized by A.Perelli, C.Viola, A.Zaccagnini and U.Zannier.

1 April to 31 July 2005, Diophantine Geometry, Mathematical Research Center Ennio De Giorgi, Scuola Normale Superiore, Pisa, Italy. Organized by Y. Bilu, E. Bombieri, D. Masser, L. Szpiro and U. Zannier.

June 22, 2007, La journée de Mathan, Bordeaux I University, Talence, France. Organized by J.Fresnel.

10 to 15 September 2007, Arithmetic Geometry, Cetraro, CIME Foundation. Organized by P.Corvaja and C.Gasbarri.

8 to 12 October 2007, Développements récents en Approximation Diophantienne, CIRM, Luminy, France. Organized by B.Adamcewski, N.Ratazzi and T. Rivoal.

6 to 10 July 2009, XXVIèmes Journées Arithmétiques, SaintEtienne, France. Scientific Committee: S.Akiyama, F.Amoroso, K.Buzzard, B.Conrad, K.Consani, P.Liardet, R.Pink, P.Tretkoff (née Cohen), J.Urbanowicz, G. Van der Geer. Organizing Committee: D.Essouabri, F.Foucault, G.Grekos, F.Hennecart, F.Pellarin, O.Robert.

4 to 8 October 2010, Number theory and its applications. An international conference dedicated to Kàlmàn Gyory, Attila Petho, Jànos Pintz, Andràs Sàrközy, Debrecen, Hungary. Organized by A.Bèrczes, A.Fazekas, K.Gyarmati, L.Hajdu and À.Pintèr.

12 to 17 september 2011, XIX Congresso dell'Unione Matematica Italiana, Bologna, Italy. Scientific Committee: F.Brezzi, L.Ambrosio, B.Franchi, M.Pulvirenti and A.Verra.

18 to 21 september 2011, 67th Séminaire Lotharingien de Combinatoire, joint session with XVII Incontro Italiano di Combinatoria Algebrica, Bertinoro, Italy. Organized by M. Barnabei, F. Bonetti, C. Krattenthaler and V. Strehl.

11 Research

Keywords: Diophantine approximation and equations, Padé and Padé-type approximations, (poly)logarithms, irrationality/non-quadraticity measures (over a number field), Riemann zeta function, hypergeometric functions.

11.1 Polynomial-exponential determinants

Let $\mathbb{K}$ be a field of characteristic 0. A polynomial-exponential equation is an equation of the form

$$\sum_{l=1}^{n} P_l(\boldsymbol{x})\boldsymbol{\alpha}_l(\boldsymbol{x}) = 0, \quad \boldsymbol{x} = (x_1, \ldots, x_s) \in \mathbb{Z}^s \tag{1}$$

where $\boldsymbol{\alpha}_l = (\alpha_{l1}, \ldots, \alpha_{ls}) \in (\mathbb{K}^\times)^s$, $\boldsymbol{\alpha}_l^{\boldsymbol{x}} = \alpha_{l1}^{x_1} \cdots \alpha_{ls}^{x_s}$ and $P_1, \ldots, P_n$ are polynomials in s variables with coefficients in $\mathbb{K}$.

The equation (1) is said to be *purely exponential* if all polinomials P_l are constants.

Let $k \geq m \geq 1$ be integers, and let $r_1, \ldots, r_m$ be positive integers such that

$$r_1 + \cdots + r_m = k. \tag{2}$$

Let $\alpha_1, \ldots, \alpha_m \in \mathbb{K}^\times$. Let $G(\boldsymbol{x}) = G(\alpha_1, \ldots, \alpha_m; r_1, \ldots, r_m; x_1, \ldots, x_k)$ be the function of integer variables $\boldsymbol{x} = (x_1, \ldots, x_k)$ defined by

$$G(\boldsymbol{x}) = \begin{vmatrix} \alpha_1^{x_1} & \cdots & x_1^{r_1-1}\alpha_1^{x_1} & \cdots & \alpha_m^{x_1} & \cdots & x_1^{r_m-1}\alpha_m^{x_1} \\ \alpha_1^{x_2} & \cdots & x_2^{r_1-1}\alpha_1^{x_2} & \cdots & \alpha_m^{x_2} & \cdots & x_2^{r_m-1}\alpha_m^{x_2} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_1^{x_k} & \cdots & x_k^{r_1-1}\alpha_1^{x_k} & \cdots & \alpha_m^{x_k} & \cdots & x_k^{r_m-1}\alpha_m^{x_k} \end{vmatrix}. \tag{3}$$

The determinant $G(\boldsymbol{x})$ is a generalization of Méray's determinant [Me] (see also [Am]).

The function G satisfies

$$G(x_1, \ldots, x_k) = (\alpha_1^{r_1} \cdots \alpha_m^{r_m})^{x_1} G(0, x_2 - x_1, \ldots, x_k - x_1). \tag{4}$$

Thus in the study of the equation

$$G(\boldsymbol{x}) = 0, \tag{5}$$

we may assume $x_1 = 0$.

From now on we assume that $\alpha_1, \ldots, \alpha_m \in \mathbb{K}^\times$ satisfy the following condition:

$$\alpha_h/\alpha_i \text{ is not a root of unity for all } h \neq i. \tag{6}$$

A subdeterminant Δ of $G(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;x_1,\dots,x_k)$ is said to be *well placed* if it is also of type (3), i.e. if there exist positive integers m', $i_1,\dots,i_{m'}$, $r'_1,\dots,r'_{m'}$, k', $j_1,\dots,j_{k'}$, with $1\le m'\le m$, $1\le i_1<\dots<i_{m'}\le m$, $1\le r'_h\le r_{i_h}$ for each $h=1,\dots,m'$, $k'=r'_1+\dots+r'_{m'}$, $1\le j_1<\dots<j_{k'}\le k$, such that

$$\Delta=G(\alpha_{i_1},\dots,\alpha_{i_{m'}};r'_1,\dots,r'_{m'};x_{j_1},\dots,x_{j_{k'}}).$$

We say that a solution $(x_1,\dots,x_k)\in\mathbb{Z}^k$ of (5) is in *general position* if no well-placed subdeterminant Δ of $G(\boldsymbol{x})$ vanishes.

If $m=k$, and hence $r_1=\dots=r_m=1$, the equation (5) becomes purely exponential. In this case, we put

$$F(\boldsymbol{x})=G(\alpha_1,\dots,\alpha_m;1,\dots,1;x_1,\dots,x_m).$$

We have

Theorem 11.1. (Schlickewei and Viola [Sc-Vi3]) *Let $\alpha_1,\dots,\alpha_k\in\mathbb{K}^\times$ satisfy (6). Then the equation*

$$F(0,y_2,\dots,y_k)=0 \tag{7}$$

has at most $\exp((6k!)^{3k!})$ solutions $(y_2,\dots,y_k)\in\mathbb{Z}^{k-1}$ in general position.

It is natural to ask whether and how we can extend theorem 11.1 to the equation $G(0,y_2,\dots,y_k)=0$, if $1<m<k$. The following theorem answers this question in the case where $\mathbb{K}$ is a number field and $k=4$:

Theorem 11.2. (Marcovecchio [Ma1]) *Let $\mathbb{K}$ be a number field of degree d, and let $\alpha_1,\dots,\alpha_m\in\mathbb{K}^\times$, with $2\le m\le 3$, satisfy (6). Then the equation*

$$G(0,y_2,y_3,y_4)=0$$

has at most

$$2^{2^{25}}d^{38400}$$

solutions in general position.

The proof of this theorem uses arguments occurring in the proof of Theorem 11.1, combined with a combinatorial analysis of certain special cases.

11.2 $\mathbb{Q}(\alpha)$–linear independence of $1, \mathrm{Li}_1(\alpha), \mathrm{Li}_2(\alpha)$

The polylogarithm $\mathrm{Li}_s(z)$ is defined, for $z \in \mathbb{C}$, $|z| < 1$ and any positive integer s, by the following series:

$$\mathrm{Li}_s(z) := \sum_{k=1}^{+\infty} \frac{z^k}{k^s}. \tag{8}$$

In an unpublished part of my doctoral thesis I proved the linear independence over $\mathbb{Q}(\alpha)$ of 1, $\mathrm{Li}_1(\alpha)$, $\mathrm{Li}_2(\alpha)$ for a suitable class of algebraic numbers α.

For this purpose I used Padé approximants of type II of the functions $1, \mathrm{Li}_1(z), \mathrm{Li}_2(z)$, which have been introduced by Hata [H1]. My generalization of Hata's results is similar to the extension given in [Am-Vi] of the method introduced in [V].

The following examples illustrate the results obtained:

$$1, \mathrm{Li}_1(\alpha), \mathrm{Li}_2(\alpha) \text{ are } \mathbb{Q}(\alpha)\text{–linearly independent}$$

if:

1. $\alpha = p/q$ is rational.

$$\begin{array}{lll} p = 1, & q \leq -8 \text{ or } q \geq 12 & \text{(Hata [H1])} \\ p = 2, & q \leq -65 \text{ or } q \geq 79 & \\ p = 3, & q \leq -218 \text{ or } q \geq 250 & \\ p = 4, & q \leq -517 \text{ or } q \geq 573, & \end{array}$$

more generally, $p > 0$ and $|q| > \widetilde{q}(p)$, where one asimptotically has

$$\widetilde{q}(p) \sim \frac{4e^4}{27} p^3,$$

for $p \to +\infty$. Here and in the rest of this section p and q are integers.

2. α is quadratic imaginary, $\alpha = i\sqrt{1/q}$, with $q \geq 108$, and therefore in particular $\alpha = i/s$, with $|s| \geq 11$. More generally $\alpha = i\sqrt{p/q}$, with $p > 0$ and $q \geq \widetilde{q}(p) > 0$, with the asymptotic behaviour

$$\widetilde{q}(p) \sim \left(\frac{4e^4}{27}\right)^2 p^3,$$

for $p \to +\infty$.

3. α is quadratic real. $\alpha = 577 - 408\sqrt{2}$, $362 - 209\sqrt{3}$, $682 - 305\sqrt{5}$.

4. α is cubic complex. We can choose α equal to one of the two complex conjugate roots of $x^3 + qx^2 + 1$, with $q \geq 5381$, or of $x^3 + qx^2 - 1$, with $q \leq -4845$.

More generally, α may be chosen among the roots of $x^3 + qx^2 + p$, with $pq > 0$, and $|q| > \widetilde{q}(p)$, where

$$\widetilde{q}(p) \sim \left(\frac{4e^6}{27}\right)^2 |p|^3.$$

for $p \to +\infty$.

5. α cubic real, α equals the real root of $x^3 + qx + 1$, close to $-1/q$, with $q \geq 69008$, or $q \leq -60744$.

6. The degree $D \geq 3$ of α is arbitrary. We may choose α equal to one of the two roots of $x^D + qx^2 + 1$ close to $\pm i/\sqrt{q}$, with $q > \widetilde{q}(D)$, where

$$\widetilde{q}(D) \sim \left(\frac{4e^{2D}}{27}\right)^2$$

for $D \to +\infty$.

We can also choose α equal to the real root of $x^D + qx + 1$ close to $-1/q$, with $|q| > \widetilde{q}(D)$, where

$$\widetilde{q}(D) \sim \frac{4e^{4D}}{27}$$

for $D \to +\infty$.

11.3 Linear independence of linear forms in polylogarithms

We prove in [Ma2] that for any non-zero algebraic number α such that $|\alpha| < 1$, the vector space over $\mathbb{Q}(\alpha)$ spanned by $1, \mathrm{Li}_1(\alpha), \mathrm{Li}_2(\alpha), \ldots$ has infinite dimension, where $\mathrm{Li}_s(z)$ is the function defined in (8).

In the special case where α is rational, this result is due to Rivoal [Ri]. In [Ma2] we avoid the saddle point method in several variables using the determinant method, introduced by Siegel and used in this context by Nikishin [N]. Our generalization uses a result due to Fischler and Rivoal [Fi-Ri]. Furthermore, it is analogous to the extension due to Amoroso and Viola [Am-Vi] of the method of Viola [V].

More specifically, we use the hypergeometric function

$$N_n(a,q,r;z) = \frac{n!^{a-r}(rn)!^{a+1}}{((r+1)n)!^{a-q}((r+1)n+1)!^q}\, z^{-rn-1}\times$$
$$\times \sum_{s\geq 0} \frac{(rn+1)_s^{a+1}}{((r+1)n+1)_s^{a-q}((r+1)n+2)_s^q}\, \frac{z^{-s}}{s!},$$

where $0 \leq q \leq a$, $z \in \mathbb{C}$, $|z| > 1$ $n \in \mathbb{N}$ and $0 < r < a$ is a real parameter. One has the decomposition

$$N_n(a,q,r;z) = \sum_{h=1}^{a} A_n^{(h)}(a,q,r;z)\mathrm{Li}_h(1/z) - A_n^{(0)}(a,q,r;z), \tag{9}$$

where $A_n^{(h)}(a,q,r;z)$ is a polynomial in z with rational coefficients. Our main result is represented by the formula

$$\det(A_n^{(h)}(a,q,r;z))_{\substack{h=0,\ldots,a\\ q=0,\ldots,a}} = \pm\left(\frac{(rn)!}{n!^r}\right)^{a+1}(z-1)^{an-rn-1}.$$

This proves that the linear forms (9), for $q = 0,1,\ldots,a$, are linearly independent for any $z \neq 1$.

11.4 The Rhin-Viola method for $\log 2$

In [Ma3] we use the method of Rhin and Viola [Rh-Vi1], [Rh-Vi2], [Rh-Vi3] to improve the non-quadraticity measures of logarithms of rational numbers proved by Hata [H2]. E.g. we prove that

$$\mu_2(\log 2) < 15.6515.$$

By the same method we improve the irrationality measure of $\log 2$ due to Rukhadze [Ru], obtained in 1987.

In fact, we prove that for any rational number p/q with sufficiently large q we have

$$\left|\log 2 - \frac{p}{q}\right| > q^{-3.57455391},$$

i.e.

$$\mu(\log 2) < 3.57455391.$$

The previous record in [Ru] was

$$\mu(\log 2) < 3.89139978.$$

Our improvement uses a family of double complex integrals of certain rational functions, namely

$$I(h,j,k,l,m,q;x) := x^{\max\{0,q-l,m-h\}}(1-x)^{k+l+m+1} \times \int_{s=0}^{i\infty}\int_{t=0}^{-i\infty} \frac{s^h t^j \, dt \, ds}{(1-s)^{l+k-j+1}(s-t)^{h+j-k+1}(t-x)^{k+m-h+1}}, \tag{10}$$

where h,j,k,l,m,q are positive integers such that $h+j+q=k+l+m$ and that $l+k-j, h+j-k, k+m-h$ are also positive, x is a real parameter satisfying $0<x<1$. The shape of this function is suggested by the construction introduced by Sorokin [So] of simultaneous Padé approximants to $1, \log x, \ldots, \log^k x$ at the point $x=1$.

A tool used in the proof is the"discrete Laplace method", in the form developed by Ball and Rivoal [Ba-Ri], which allows us to obtain the asymptotic behavior of the coefficient of $\log 2$ in the imaginary part of the integral (10), where one sets $x=1/2$. A weak version of the saddle point method in $\mathbb{C}^2$ due to Hata yields the asymptotic upper bound of the integral. The permutation group method due to Rhin and Viola is used to find good denominators for the coefficients of 1 and $\log 2$ in the imaginary part of the integral, and of 1, $\log 2$ and $\log^2 2$ in its real part.

11.5 Logarithms of algebraic numbers

This is a joint work with C.Viola. In a paper [Ma-Vi] to appear in Journal of Australian Mathematical Society we study Diophantine properties of logarithms of certain algebraic numbers. We introduce, for a complex parameter x not 0 or 1, two double integrals instead of a single one. If $x \in \mathbb{R}^+$, these are the integral (10) and its complex conjugate. Then we choose x to be an algebraic number α, or one of its algebraic conjugates, both in the two integrals mentioned above and in the associated double contour integral. We thus obtain, by an application of the $\mathbb{C}^2$–saddle point method, the required asymptotic behaviours of our approximations to $\log \alpha$ and its square. Then we apply an improved version of an irrationality criterion due to [Am-Vi], and a generalization of that criterion to the dimension 2.

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