PRESENTATION
of R. Marcovecchio's article
"The Rhin-Viola method for $\log 2$"

In this report I would like to express my opinion about recent results of R. Marcovecchio published in 2009 in Acta Arithmetica, v. 139, no. 2, pp. 147-184. The main results of this article are two new upper bounds for irrationality measure μ_1 and non-quadraticity measure μ_2 of the number $\log 2$. By definition μ_1 is the number satisfying the following condition:

for any $\varepsilon > 0$ the inequality $|\log 2 - p/q| < q^{-\mu_1+\varepsilon}$ has infinitely many solutions $p, q \in \mathbb{Z}$ and the inequality $|\log 2 - p/q| < q^{-\mu_1-\varepsilon}$ has only a finite set of solutions $p, q \in \mathbb{Z}$.

The definition of μ_2 is analogous. The only difference is that the rational approximations are replaced by quadratic irrational approximations U, and the denominator q is replaced by the height of U.

Naturally these definitions can be given not only for $\log 2$ but for any real number α. For example, the equality $\mu_1(e) = 2$ belongs essentially to Euler. Any algebraic number α has $\mu_1(\alpha) = 2$, K. Roth, 1954. The values $\mu_1(\pi)$ and $\mu_1(\log 2)$ are not known. It is not difficult to prove that $\mu_1(\alpha) \geq 2$ for any real irrational α, Dirichlet. Upper bounds for $\mu_1(\pi)$ were proved by

$\mu_1(\pi) \leq 42$, *K. Mahler*, 1954, $\mu_1(\pi) \leq 20$, *M. Mignotte*, 1974,
$\mu_1(\pi) \leq 8.0161\ldots$, *M. Hata*, 1993, $\mu_1(\pi) \leq 7.61\ldots$, *V. Salikhov*, 2008.

The study of arithmetical properties of classical constants like e, π and others is a traditional subject in the theory of diophantine approximations. The logarithm $\log 2$ presents the family of logarithms of algebraic numbers. It is a natural model for comparison of different methods used for evaluation of the measure of irrationality for logarithms of algebraic numbers.

In his Acta Arithmetica article R. Marcovecchio proves the best up today bound

$$\mu_1(\log 2) \leq 3.574\ldots. \tag{1}$$

The previous record was proved in 1987 by E. Rukhadze: $\mu_1(\log 2) \leq 3.891\ldots$. After many attempts this result was improved only in Marcovecchio's article.

The main new ingredient introduced by Marcovecchio is a new and absolutely unexpected construction of rational approximations to $\log 2$. It is based on double complex integrals

$$I_n(x) = x^n(1-x)^{15n+1} \int_{s=0}^{i\infty} \int_{t=0}^{-i\infty} \frac{s^{5n}t^{6n}\,dt\,ds}{(1-s)^{3n+1}(s-t)^{7n+1}(t-x)^{5n+1}}, \qquad n \geq 1. \tag{2}$$

By a very tricky way Marcovecchio proves that

$$\frac{1}{\pi}\Im I_n(x) = P_n(x)\log(1/x) - Q_n(x), \qquad x \in \mathbb{R},$$

where $P_n(x)$ and $Q_n(x)$ are polynomials. The coefficients of $P_n(x)$ are integers, and the coefficients of $Q_n(x)$ are rational numbers. Moreover $\deg P_n(x) \leq 11n$ and $\deg Q_n(x) \leq 11n$.

The next rather difficult arithmetical step in the proof is a very precise upper bound for the common denominator of coefficients of the polynomial $Q_n(x)$. Marcovecchio proves that for

$$d_m = \text{l.c.m.}[1,2,\ldots,m] \qquad \text{and} \qquad \Delta_n = \prod_{\{n/p\}\in\Omega} p,$$

where $\Omega = [1/6, 3/7)\bigcup[1/2, 5/7)\bigcup[3/4, 6/7)$, the polynomial $\dfrac{d_{7n}}{\Delta_n} Q_n(x)$ has integer coefficients. The factor Δ_n here is a common divisor of all integer coefficients of the polynomial $P_n(x)$. To prove this property Marcovecchio uses the powerful method introduced in 1996 by G. Rhin and C. Viola to study approximation properties of the number $\zeta(2)$. This so called group method was applied later by G. Rhin and C. Viola to the number $\zeta(3)$. In both cases it gave the best up today upper bounds for the irrationality measures of these numbers. The idea is to find a transformation of parameters in the integral construction of approximations that transform the integral into another one of the same form but having another set of parameters and an integer factor before the integral. These transformations form a group and a very delicate consideration of factors before the integrals allow to construct the common divisor of all coefficients. The advantage of Marcovecchio's construction is a comparatively simple description of the group of transformations for the double complex integral. Lot of computer calculations was used to determine the set Ω. The common divisor Δ_n of coefficients of $P_n(x)$ calculated in the article cannot be improved.

The next step in the proof is a calculation of the asymptotic for $\log|P_n(x)|$ and an upper bound for $\log|I_n(x)|$ for n tending to infinity. The proof of this asymptotic is based on the representation

$$P_n(x) = (1-x)^{15n+1} \sum_{r\geq 2n} \binom{r+3n}{3n}\binom{r+5n}{7n}\binom{r+4n}{5n} x^r.$$

To prove an upper bound for $\log|I_n(x)|$ Marcovecchio uses the $\mathbb{C}^2$ saddle point method to the double complex integral. And this is very technical work. Finally he takes $x = 1/2$ and with the help of Hata's lemma proves the upper bound (1).

The representation

$$\Re I_n(x) = P_n(x)\frac{1}{2}\log^2(1/x) - Q_n(x)\log(1/x) + R_n(x),$$

where $R_n(x) \in \mathbb{Q}[x]$, is used to estimate $\mu_2(\log 2)$. Calculation of common denominator for coefficients of $R_n(x)$ and another choice of parameters in the integral (2) lead to the bound for the non-quadraticity measure $\mu_2(\log 2) \leq 15.651\ldots$ that essentially improves the last result of Hata $\mu_2(\log 2) \leq 25.046\ldots$ proved in 2000.

I highly evaluate results proved by R. Marcovecchio in his Acta Arithmetica article. He introduced some elegant new ideas in this rather technical branch of diophantine approximations theory, adopted in clever way some classical methods to his situation and essentially improved some old and well known results. I believe that Marcovecchio started by his article a new line in effective constructions of diophantine approximations to values of polylogarithmic functions.

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