Curriculum vitae

Gabriele Ranieri

Personal

Date of birth	April 11, 1979
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Past and current positions

September 1, 2007 - August 31, 2008	A.T.E.R. (University of Caen, France).
September 1, 2008 - August 31, 2009	A.T.E.R. (University of Caen, France).
September 1, 2009 - August 31, 2010	Post-doc (University of Basel, Switzerland).
October 1, 2010 - December 31, 2010	Post-doc (University of Goettingen, Germany).

University cursus

2007	P.H.D. thesis. Mention: très honorable (very good).
2004-2007	P.H.D. in mathematics at Pisa University (Italy). Since 2005 P.H.D. in mathematics at Caen University (France) and "Cotutelle" between the universities of Pisa and Caen.
2003	Laurea (master's degree) at Pisa University (Italy). Mention: 110/110 cum laude.
1998-2003	Student at Pisa University.

1 Research

1.1 Thesis

Original title	Rang de l'image du groupe des unités; conjecture de Bremner.
English title	Rank of the image of the units group; Bremner's conjecture.
Thesis advisors	Francesco AMOROSO (University of Caen), Roberto DVORNICICH (University of Pisa)

1.2 Articles

1. G. RANIERI, *Rang de l'image du groupe des unités et polynômes lacunaires*, Acta Arithmetica **127**, 87-96 (2007). (Rank of the image of units group and lacunary polynomials).

   ABSTRACT. Amoroso [Amo] introduced a new class of number fields, the so-called class of PCM-fields (proche d'un corps CM), having interesting arithmetic properties. The main result of our work is the proof of the fact that no totally real field is PCM. Moreover, we show how our proof, in the special case of finite totally real abelian extensions of $\mathbb{Q}$, implies a lower bound for the number of non-zero coefficients of polynomials, whose certain zeros are roots of 1.

2. G. RANIERI, *Générateurs de l'anneau des entiers d'une extension cyclotomique*, Journal of Number Theory **128**, 1576-1586 (2008). (Power integral bases of a cyclotomic extension).

   ABSTRACT. Let p be an odd prime and $q = p^m$, where m is a positive integer. Let ζ_q be a primitive qth root of 1 and $\mathcal{O}_q$ be the ring of integers of $\mathbb{Q}(\zeta_q)$. In [Ga-Ro], I. Gaál and L. Robertson showed that if $(h_q^+, p(p-1)/2) = 1$ (where h_q^+ is the class number of $\mathbb{Q}(\zeta_q + \overline{\zeta_q})$) then, if $\alpha \in \mathcal{O}_q$ is a generator of $\mathcal{O}_q$ either α is equal to a conjugate of an integer translate of ζ_q or $\alpha + \overline{\alpha}$ is an odd integer. In this paper we show that we can remove the hypothesis over h_q^+. In other words we prove that if α is a generator of $\mathcal{O}_q$, then either α is a conjugate of an integer translate of ζ_q or $\alpha + \overline{\alpha}$ is an odd integer.

3. G. RANIERI, *Power bases for rings of integers of abelian imaginary fields*, accepted on the Journal of the London Mathematical Society.

   ABSTRACT. Let L be a number field and let $\mathcal{O}_L$ be its ring of integers. It is a very difficult problem to decide whether $\mathcal{O}_L$ has a power basis. Moreover, if a power basis exists, it is hard to find all the generators of $\mathcal{O}_L$ over $\mathbb{Z}$. In this paper, we show that if α is a generator of the ring of integers of an abelian imaginary field whose conductor is relatively prime to 6, then either α is an integer translate of a root of unity, or $\alpha + \overline{\alpha}$ is an odd integer. From this result and other remarks it follows that if β is a generator of the ring of integers of an abelian imaginary field with conductor relatively prime to 6 and β is not an integer translate of a root of unity, then $\beta\overline{\beta}$ is a generator of the ring of integers of the maximal real field contained in $\mathbb{Q}(\beta)$.

   Finally, if $d > 1$ is an integer relatively prime to 6, we prove, using the main result of Gras [Gra], that all but finitely many abelian imaginary extensions of $\mathbb{Q}$ of degree $2d$ have a ring of integers that does not have a power basis.

4. B. ANGLÈS, G. RANIERI, *On the linear independence of p-adic L functions modulo p*, accepted on Ann. Inst. Fourier, Grenoble.

   ABSTRACT. Let $p \geq 3$ be a prime. Let $n \in \mathbb{N}$ be such that $n \geq 1$, let $\chi_1, \ldots, \chi_n$ be characters of conductor d not divisible by p and let ω be the Teichmüller character. For all i between 1 and n, for all j such that $0 \leq j \leq (p-3)/2$, set

$$\theta_{i,j} = \begin{cases} \chi_i \omega^{2j+1} & \text{if } \chi_i \text{ is odd;} \\ \chi_i \omega^{2j} & \text{if } \chi_i \text{ is even.} \end{cases}$$

Let K be the smallest extension of $\mathbb{Q}_p$ that contains all the values of the characters χ_i and let π be a prime of the valuation ring $\mathcal{O}_K$ of K. For all i, j let $f(T, \theta_{i,j})$ be the Iwasawa series associated to $\theta_{i,j}$ and $\overline{f(T, \theta_{i,j})}$ its reduction modulo (π). Finally let $\overline{\mathbb{F}_p}$ be an algebraic closure of $\mathbb{F}_p$. Our main result is that if the characters χ_i are all distinct modulo (π), then 1 and the series $\overline{f(T, \theta_{i,j})}$, for $1 \leq i \leq n$ and $0 \leq j \leq (p-3)/2$, are linearly independent over a certain field Ω that contains $\overline{\mathbb{F}_p}(T)$. In the case $d = 1$, the linear independence of the series $\overline{f(T, \theta_{i,j})}$ seems to confirm the hypothesis that Ferrero and Washington made to build an heuristic to determine a bound for the λ-invariant of $\mathbb{Q}(\zeta_p)$.

1.3 Summary of the thesis

Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of number fields, let Γ_n be the group of the $\mathbb{Q}$-automorphisms of K_n. Set $d_n = [K_n : \mathbb{Q}]$ and E_{K_n} the group of the units of the ring of integers of K_n. Suppose that

$$\lim_{n\to+\infty} d_n = +\infty.$$

We say that the sequence $\{K_n\}_{n\in\mathbb{N}}$ is a family of fields near a CM-field (PCM-fields), if there exist some non-zero elements $\phi_n \in \mathbb{Z}[\Gamma_n]$ such that:

$$\lim_{n\to+\infty} \frac{\|\phi_n\|_1 \max\{r_{\phi_n}, 1\}}{d_n} = 0,$$

where $\|\phi_n\|_1$ is the size of ϕ_n and r_{ϕ_n} is the rank over $\mathbb{Z}$ of

$$E_{K_n}^{\phi_n} = \{\beta \in E_{K_n} \text{ s.t. } \exists \alpha \in E_{K_n} \text{ s.t. } \beta = \alpha^{\phi_n}\}.$$

In the first section of our thesis we prove that all family of totally real extensions of $\mathbb{Q}$ is never PCM (see [Ran1]). Moreover we show that, in the case of a family of finite real abelian extensions of $\mathbb{Q}$, this result implies a lower bound for the number of non-zero coefficients of polynomials whose certain zeros are roots of 1 (see [Ran1]). Finally we give some necessary conditions for the existence of families of PCM fields in the general case.

Let n be a positive integer and let ζ_n be an nth primitive root of unity in $\mathbb{C}$. It is well known that the ring of integers of $\mathbb{Q}(\zeta_n)$ has a power basis and it is equal to $\mathbb{Z}[\zeta_n]$. It is an interesting problem to determine all the generators for $\mathbb{Z}[\zeta_n]$. Consider the following equivalence relation among generators of $\mathbb{Z}[\zeta_n]$: let $\alpha \in \mathbb{Z}[\zeta_n]$ be such that $\mathbb{Z}[\alpha] = \mathbb{Z}[\zeta_n]$; we say that $\beta \in \mathbb{Z}[\zeta_n]$ is equivalent to α ($\alpha \sim \beta$) if there exist $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ and an integer l such that $\beta = \pm\sigma(\alpha) + l$.

Let $p \geq 2$ be a prime number and consider $\mathbb{Z}[\zeta_p]$. In this case we know two classes of generators for this ring: the class of ζ_p and the class of $\omega = 1/(\zeta_p + 1)$. Bremner [Bre] conjectured that there are no other classes of generators of $\mathbb{Z}[\zeta_p]$. Robertson [Rob] gave a partial answer to Bremner's conjecture: she proved that if α is a generator of $\mathbb{Z}[\zeta_p]$, then either α is equivalent to ζ_p or $\alpha + \overline{\alpha}$ is an odd integer.

It is possible to generalize Bremner's conjecture to the case of $\mathbb{Z}[\zeta_q]$, where q is the power of a prime number. In 2006 Robertson and Gaál ([Ga-Ro]) proved that if α is a generator of $\mathbb{Z}[\zeta_q]$ then, if we have $(h_q^+, p(p-1)/2) = 1$, where h_q^+ is the class number of $\mathbb{Q}(\zeta_q + \overline{\zeta_q})$, either α is equivalent to ζ_q or $\alpha + \overline{\alpha}$ is an odd integer.

In the second section of our thesis we improve the main result of [Ga-Ro]. Indeed we show that the hypothesis on h_q^+ can be removed. In other words we prove that if α is a generator of $\mathbb{Z}[\zeta_q]$, then either α is equivalent to ζ_q or $\alpha + \overline{\alpha}$ is an odd integer (see [Ran2]). Moreover we generalize the ideas of [Ran2] to study (not totally) the case of CM fields.

Let L/K be a cyclic extension of number fields such that $[L : K] = D \geq 2$; let $\mathcal{O}_K$, $\mathcal{O}_L$ be respectively the rings of algebraic integers of K and L. Finally let σ be a generator of $\mathrm{Gal}(L/K)$ and $\gamma \in \mathcal{O}_L$. For al integer $i \geq 0$ set:

$$u_i = \gamma\sigma(u_{i-1}) + 1,$$

with $u_0 = 0$. Hence $u_1 = 1$ and

$$u_i = \gamma^{1+\sigma+\cdots+\sigma^{i-2}} + \ldots + \gamma^{1+\sigma} + \gamma + 1,$$

for all i. We say that γ is a σ-hyperexceptional unit if and only if:

1. $\gamma \in \mathcal{O}_L^*$;

2. $u_2, \ldots, u_{D-1}$ are units;

3. $u_D = 0$.

We say that γ is an hyperexceptional unit if there exists a generator $\sigma \in \mathrm{Gal}(L/K)$ such that γ is σ-hyperexceptional.

Let n be a positive integer and let $\omega_1, \omega_2, \ldots, \omega_n$ be in $\mathcal{O}_L$. We say that the set $\{\omega_i\}_{1 \leq i \leq n}$ is an exceptional sequence if and only if $\omega_i - \omega_j \in \mathcal{O}_L^*$ for all $i \neq j$. Moreover we call Lenstra's constant $M(L)$ of the field L the following number:

$$M(L) = \max\{m \in \mathbb{Z} \text{ s.t. } \exists \omega_1, \ldots, \omega_m \in \mathcal{O}_L \text{ s.t. } \omega_i - \omega_j \in \mathcal{O}_L^* \; \forall 1 \leq i < j \leq m\}$$

In [Len] Lenstra proved that if $M(L) \geq C$, where C is a constant depending on $[L : \mathbb{Q}]$ and the discriminant of L, then L is an euclidean field. This result allowed Lenstra finding 132 new euclidean fields. Afterwards, other mathematicians developed methods to build exceptional sequences. Then, using Lenstra's result, they sometimes found new euclidean fields. For instance Mestre [Mes] constructed exceptional sequences using points of finite order on some elliptic curves.

The most interesting property of a σ-hyperexceptional unit γ is that the sequence $\{u_0, u_1, \ldots, u_{D-1}\}$ is an exceptional sequence. Hence, if a field L contains a σ-hyperexceptional unit, then

$$M(L) \geq D.$$

In the third section of our thesis we construct some examples of hyperexceptional units that allow us improving some known lower bounds for Lenstra's constant.

2 Teaching experience

1. TD Approfondissement en mathématiques – September 2007- January 2008 – University of Caen.

2. TD L1 Analyse – February-May 2008 – University of Caen.

3. TD L1 Algèbre – February-May 2008 – University of Caen.

4. Cours-TD L1 Analyse – February-May 2009 – University of Caen.

5. TD Master Galois Theory – September 2009-January 2010 – University of Basel.

6. TD Master Commutative Algebra – March-June 2010 – University of Basel.

3 Seminars of the number theory group of the University of Caen

1. *Groupe de travail p-adique* – 2006-2007.
   Study of the book: J. Coates et R. Sujatha, *Cyclotomic fields and Zeta Values*, Springer, 2006.

2. *Groupe de travail p-adique* – 2007-2008.
   Study of the book: H. Hida, *Elementary theory of L-functions and Eisenstein series*, London Math. Soc. Student Text 26, 1993.
   Talk : Les fonctions L de Shintani.

3. *Groupe de travail p-adique* – 2008-2009.
   Study of the book: J. Fontaine, Y. Ouyang *Theory of p-adic Galois Representations*, Springer.
   Talk: Vecteurs de Witt et anneaux de Cohen.

4. *Groupe de travail sur les points de torsions et les polynômes lacunaires* – Caen (L.M.N.O.) – March-May 2009.
   Talk: Sur les sommes de racines de l'unité dont la valeur est 0.

4 Talks and conferences

1. *Conference Sa conjectura de Catalan* – Menorca – September 2004.
   Talk: Plus argument semisimple.
2. *Trimestre di geometria diofantea* – Pisa (Scuola Normale) – March-June 2005.
3. *XVII^e rencontres arithmétiques de Caen* –« Approximation diophantienne » –June 2006.
4. *Colloque Approximation diophantienne et nombres trascendants* – Luminy, Marseille (C.I.R.M.) – September 2006.
   Talk: Une minoration du nombre de coefficients non nuls de polynômes s'annullant en certaines racines de l'unité.
5. *Seminaires du groupe de théorie des nombres de Caen* – Caen (L.M.N.O.) – October 13, 2006.
   Talk: Générateurs de l'anneau des entiers d'une extension cyclotomique.
6. *Workshop Diophantische Approximationen* – Oberwolfach (M.F.O.) – April 2007.
   Talk: Power integral bases for prime-power cyclotomic fields.
7. *Workshop Diophantine equations* – Leiden (Lorentz Center) – May 2007.
8. *Journées arithmétiques* – Edimbourgh – July 2007.
   Talk: Power integral bases for prime-power cyclotomic fields.
9. *Seminaires du groupe de théorie des nombres de Bordeaux* – Bordeaux – November 9, 2007.
   Talk: Générateurs de l'anneau des entiers d'une extension cyclotomique.
10. *XIX^e rencontres arithmétiques de Caen* –« Arithmétique des corps de fonctions en caractéristique positive » – June 2008.
11. *Iwasawa 2008* – Kloster Irsee – June-July 2008.
12. *Seminaires du groupe de théorie des nombres de Besançon* – Besançon – March 12, 2009.
    Talk: Indépendence linéaire des fonctions L p-adiques modulo p.
13. *XX^e rencontres arithmétiques de Caen* –« Cohomologie p-adique » – June 2009.
14. *Journées arithmétiques* – Saint-Etienne – July 2009.
    Talk: Power integral bases for abelian imaginary fields.
15. *Number theory seminar, Basel* – Basel – October, 8, 2009.
    Talk: On the linear independence of p-adic L-functions modulo p.
16. *Séminaires du groupe de travail p-adique Université de Caen* – à Caen (L.M.N.O.) – February, 12, 2010.
    Talk: Méthodes diophantiennes en théorie d'Iwasawa.
17. *Séminaires du groupe de théorie des nombres de Bordeaux* – Bordeaux – May, 28, 2010.
    Talk: Indépendence linéaire des fonctions L p-adiques modulo p.

References

[Amo] FRANCESCO AMOROSO, *Groupes des classes de corps proches d'un corps CM*, Preprint (2005).

[Ang] BRUNO ANGLÈS, *On the p-adic Leopoldt transform of a power series*, Acta Arith. **134**, 349-367, (2008).

[Bre] ANDREW BREMNER, *On power bases in cyclotomic number fields*, J. of Number Theory **28**, 288-298, (1988).

[Ga-Ro] ISTVÀN GAÁL, LEANNE ROBERTSON, *Power integral bases in prime-power cyclotomic fields*, J. of Number Theory, **120**, 372-384, (2006).

[Gra] MARIE NICOLE GRAS, *Non monogénéité de l'anneau des entiers de certaines extensions abéliennes de* $\mathbb{Q}$, Publ. Math. Sci. Besançon, Théor. Nombres, (1984).

[Len] HENDRIK LENSTRA, *Euclidean number fields of large degree*, Invent. Math. **36**, 237-254, (1977).

[Mes] JEAN-FRANÇOIS MESTRE, *Corps euclidiens, unités exceptionnelles et courbes elliptiques*, J. of Number Theory **13**, 123-137, (1981).

[Ran1] GABRIELE RANIERI, *Rang de l'image du groupe des unités et polynômes lacunaires*, Acta Arithmetica **127**, 87-96, (2007).

[Ran2] GABRIELE RANIERI, *Générateurs de l'anneau des entiers d'une extension cyclotomique*, J. of Number Theory **128**, 1576-1586, (2008).

[Ran3] GABRIELE Ranieri, *Power bases for rings of integers of abelian imaginary fields*, to appear on the Journal of the London Math. Soc.

[Rob] LEANNE ROBERTSON, *Power bases for cyclotomic integer rings*, J. of Number Theory **69**, 98-118, (1998).