Fractional Cauchy problems on $\mathbb{S}_1^2$ and coordinates changed random fields

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Abstract

We present some results about coordinates changed random fields and characterize the corresponding angular power spectrum. Thus, we provide a probabilistic interpretation of the anisotropies of cosmic microwave background (CMB) radiation in the theory of Big Bang.

We first introduce the fractional Cauchy problem on compact manifolds $\mathcal{M}$ and therefore specialize such result on the unit sphere $\mathcal{M} = \mathbb{S}_1^2$. The unique strong solution to the fractional Cauchy problem is the transition law of a time changed Brownian motion on $\mathbb{S}_1^2$ started at $x \in \mathbb{S}_1^2$, say $\mathfrak{B}^x(\mathfrak{L}_t^\nu)$, $t > 0$, where $\nu \in (0,1)$ is the order of the time-fractional derivative and of the new clock $\mathfrak{L}_t^\nu$ which is an inverse to a stable subordinator. We consider a random field on the sphere, say $T(x)$, $x \in \mathbb{S}_1^2$ and its spectral representation obtained by means of spherical harmonics. Thus, we define a coordinates changed random field on the sphere given by $T(\mathfrak{B}^x(\mathfrak{L}_t^\nu)) = T(x + \mathfrak{B}^{x_N}(\mathfrak{L}_t^\nu))$ where x_N is the north pole. Our new random field is characterized by a new form of the angular power spectrum. In particular, it exhibits long range dependence structure due to the non-Markovian coordinates change. Moreover, we consider the subordinated Brownian motion $\mathfrak{B}^x(\mathfrak{H}_t^\alpha)$ as coordinates change of T. In this case we obtain new random fields with short range dependence but interesting form of the new angular power spectrum. Interesting results are also obtained by considering $T(x + \mathfrak{B}^{x_N}(S_t))$ where S_t is a subordinator with symbol Ψ.

CMB radiation is a thermal radiation filling the universe almost uniformly and can be affected by some anisotropies. In the analysis of the CMB radiation a key role is played by the angular power spectrum of the observed data, that is a realization of a random field on the sphere.

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