Nonlinear Random Perturbations of PDEs: Toward an Infinite-Dimensional Theory

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based on joint work with G. Tessitore

Abstract

In finite dimensions, quasi-linear parabolic equations are classically linked to stochastic differential equations whose diffusion coefficient depends on the solution itself. In [1], Freidlin and Koralov analyzed quasi-linear parabolic equations with a small parameter multiplying the second-order term and obtained a detailed description of the long-time behavior of solutions via large deviation principles for the associated diffusions.

In this work, we present an infinite-dimensional counterpart of this theory for dynamical systems generated by partial differential equations. We consider general evolution equations in a separable Hilbert space H, driven by a linear operator A generating a strongly continuous semigroup and a nonlinear drift term b. We introduce nonlinear small random perturbations in which the diffusion coefficient depends on the solution of the associated quasi-linear Kolmogorov equation. This leads to a stochastic PDE with a nonlinear feedback structure, which can also be interpreted as an infinite-dimensional forward–backward system.

A primary difficulty is the absence of a general Hilbert-space framework for quasi-linear Kolmogorov equations. In [2], we address this issue by treating the nonlinear diffusion term as a perturbation of an Ornstein–Uhlenbeck operator and establishing existence, uniqueness, regularity, and a maximum principle under suitable structural and smallness assumptions. Building on this, in [3], we prove well-posedness of the associated stochastic PDE in appropriate Banach spaces, allowing for non-Lipschitz reaction terms with possibly unbounded growth and more general nonlinear diffusion structures.

Using the weak convergence approach, we establish a large deviation principle for the family of trajectories in $C([0,T];H)$. The corresponding rate function is characterized through a deterministic controlled equation involving a nonlinear feedback term inherited from the limiting dynamics.

These results represent initial steps toward extending the Freidlin–Koralov theory to infinite-dimensional PDEs. They provide a first systematic framework for analyzing nonlinear small random perturbations of PDEs and their long-time asymptotic behavior.

References

[1] M. I. Freidlin and L. Koralov, Nonlinear stochastic perturbations of dynamical systems and quasi-linear parabolic PDE's with a small parameter, *Probability Theory and Related Fields* (2010), 147, 273–301.

[2] S. Cerrai, G. Guatteri, G. Tessitore, Nonlinear random perturbations of PDEs and quasi-linear equations in Hilbert spaces depending on a small parameter. *Journal of Functional Analysis*(2024), 286(12), 1-57

[3] S. Cerrai, G. Guatteri, G. Tessitore, Nonlinear random perturbations of Reaction-Diffusion Equations, https://arxiv.org/abs/2506.17094