PATHWISE UNIQUENESS, STRONG EXISTENCE AND REGULARIZATION BY NOISE EFFECT

DAVIDE ADDONA, FEDERICA MASIERO

ABSTRACT

Existence and uniqueness of strong solutions to the stochastic differential equation

$$dx(t) = b(t, x(t))dt + \sigma(t, x(t))dw(t), \qquad t \in (0, T], \qquad x(0) = x_0 \in \mathbb{R}^d, \tag{SDE}$$

where w is a d-dimensional Brownian motion, classically hold under Lipschitz assumptions on the coefficients. In 1974, Zvonkin proved that, in the case $d = 1$, (SDE) admits a unique strong solution under the weaker assumption that b is bounded and measurable. This result was obtained by using the Yamada–Watanabe theorem, which states that if (SDE) admits a weak solution and pathwise uniqueness holds, then it also admits a strong solution. Hence, once weak existence has been established, it only remains to prove pathwise uniqueness. Zvonkin proved pathwise uniqueness by exploiting a suitable transformation, based on Kolmogorov equations and Itô's formula, which allows one to remove the irregular drift term in (SDE). This result was generalized by Veretennikov (1980) to the case $d > 1$. More general, possibly unbounded, drift terms were considered in subsequent papers (see, for instance, Krylov–Röckner (2005)). These results can also be interpreted in terms of *regularization by noise*, since, in general, the ordinary differential equation corresponding to (SDE), namely the one with $\sigma = 0$, is ill-posed when b is only bounded and measurable (recall, for instance, the classical example $\dot{x} = x^\alpha$ with $x(0) = 0$ and $\alpha \in (0, 1)$).

In infinite dimension, the starting point for similar investigations is the extension of the Yamada–Watanabe theorem to mild solutions due to Ondrejat (2004). Thus, existence of strong mild solutions follows from existence of weak solutions (usually obtained by means of Girsanov's theorem or compactness methods) together with pathwise uniqueness for such weak solutions. The first results in this direction are due to Flandoli, Gubinelli, and Priola, who in 2010 proved the well-posedness of a transport equation under a stochastic perturbation, and to Da Prato and Flandoli, who in the same year proved pathwise uniqueness for a class of infinite-dimensional stochastic differential equations modeling parabolic stochastic partial differential equations, under a Hölder continuity assumption on the drift term. In both cases, the authors used the so-called Itô–Tanaka trick, a transformation inspired by Zvonkin's method. Since then, many authors have studied stochastic partial differential equations with non-smooth drift, and different approaches, besides those described above, have also been considered.

In the first talk we introduce the main ideas used to prove pathwise uniqueness for stochastic differential equations of the form (SDE), and we discuss the main difficulties in extending these ideas to the infinite-dimensional setting.

In the second talk we present a variant of the Itô-Tanaka trick based on the use of Backward Stochastic Differential Equations (BSDEs), and we prove regularization by noise results for the stochastic wave equation; moreover we discuss how our method based on BSDEs applies also to stochastic evolution equations of parabolic type.